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Research Article

Analysis, Control, Synchronization and SPICE Implementation of a Novel 4-D Hyperchaotic Rikitake Dynamo System without Equilibrium

S. Vaidyanathan*[1], Ch. K. Volos[2], and V. -T. Pham[3]

[1]*Research and Development Centre, Vel Tech University, Avadi, Chennai-600062, Tamil Nadu, India.*
[2]*Department of Physics, Aristotle University of Thessaloniki, Thessaloniki, GR-54124, Greece.*
[3]*School of Electronics and Telecommunications, Hanoi University of Science and Technology, Hanoi, Vietnam.*

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Abstract

Chaos theory has wide applications and its importance can be seen by the voluminous publications on various applications in several branches of science, commerce and engineering. Control, tracking or regulation and synchronization of different types of chaotic systems are importance areas of research in the control literature and various methods have been adopted over the past few decades for tackling these research problems. Also, the discovery of novel chaotic and hyperchaotic systems in various applications, their qualitative properties and the control of such systems are also important research areas in chaos theory. This paper announces a novel 4-D hyperchaotic Rikitake dynamo system, which is derived by adding a state feedback control to the famous 3-D Rikitake two-disk dynamo system (1958). The frequent and irregular reversals of the Earth's magnetic field inspired a number of early studies involving electrical currents within the Earth's molten core. One of the first such models to exhibit reversals was Rikitake's two-disk dynamo system (Rikitake, 1958). This paper discusses the qualitative properties of the novel hyperchaotic Rikitake dynamo system. We note that the novel hyperchaotic Rikitake dynamo system has no equilibrium points. The Lyapunov exponents of the hyperchaotic Rikitake dynamo system are found as $L_1 = 0.09136, L_2 = 0.02198, L_3 = 0$ and $L_4 = -2.11190$. The Kaplan-Yorke fractional dimension of the novel hyperchaotic Rikitake dynamo system is found as $D_{KY} = 3.05367$. Next, this paper discusses control and synchronization of the novel hyperchaotic Rikitake dynamo system with unknown parameters using adaptive control method. The main results are established using Lyapunov stability theory and numerically illustrated using MATLAB. Finally, for the 4-D novel hyperchaotic system, an electronic circuit realization in SPICE has been described to confirm the feasibility of the theoretical hyperchaotic Rikitake dynamo model.

Keywords: Chaos, chaotic systems, hyperchaos, hyperchaotic systems, adaptive control, chaos synchronization, circuit simulation.

1. Introduction

Chaotic systems are defined as nonlinear dynamical systems which are very sensitive to initial conditions, topologically mixing and also with dense periodic orbits [1]. The first 3-D chaotic system was experimentally verified by Lorenz [2].

The sensitivity to initial conditions of a chaotic system is indicated by a positive Lyapunov exponent. If the sum of all Lyapunov exponents is negative, then the chaotic system is called a dissipative system.

Great interest has been shown in the chaos literature in the modelling of many 3-D chaotic systems such as Rössler system [3], Rabinovich system [4], ACT system [5], Sprott systems [6], Chen system [7], Lü system [8], Shaw system [9], Feeny system [10], Shimizu system [11], Liu-Chen system [12], Cai system [13], Tigan system [14], Colpitt's oscillator [15], WINDMI system [16], Zhou system [17], etc.

Recently, many 3-D chaotic systems have been discovered such as Li system [18], Elhadj system [19], Pan system [20], Sundarapandian system [21], Yu-Wang system [22], Sundarapandian-Pehlivan system [23], Zhu system [24], Vaidyanathan systems [25-33], Vaidyanathan-Madhavan system [34], Pehlivan-Moroz-Vaidyanathan system [35], Pham systems [36-37], etc.

Chaos theory has many important applications in science and engineering such as vibration control [38-40], oscillators [41-43], robotics [44-47], chemical reactors [48-50], biology [51,52], ecology [53,54], cardiology [55], memristors [56-58], neural networks [59-61], secure communications [62-65], cryptosystems [66-69], networks

* E-mail address: sundarvtu@gmail.com
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[70, 71], economics [72-74], market forecasting [75], etc. Several works have been carried out by researchers in exploring new applications of chaos theory in diverse fields.

Chaos control and chaos synchronization are important research problems in the chaos theory. In the last three decades, many mathematical methods have been developed successfully to address these research problems.

The study of control of a chaotic system investigates methods for designing feedback control laws that globally or locally asymptotically stabilize or regulate the outputs of a chaotic system.

Many methods have been developed for the control and tracking of chaotic systems such as active control [76-80], adaptive control [81-87], backstepping control [88-90], sliding mode control [91, 92], etc.

A hyperchaotic system is generally defined as a chaotic system with at least two positive Lyapunov exponents. Thus, the dynamics of a hyperchaotic system are expended in several different directions simultaneously. Thus, the hyperchaotic systems have more complex dynamical behaviour and hence they have miscellaneous applications in engineering such as secure communications [93-95], cryptosystems [96-98], encryption [99-101], electrical circuits [102-105], etc.

Chaos synchronization problem deals with the synchronization of a couple of systems called the *master* or *drive* system and the *slave* or *response* system. To solve this problem, control laws are designed so that the output of the slave system tracks the output of the master system asymptotically with time.

Because of the butterfly effect, the chaos synchronization problem is seemingly a challenging research problem even when the initial conditions of the master and slave systems are nearly identical because of the exponential divergence of the outputs of the two systems in the absence of any control. The synchronization of chaotic systems has applications in secure communications [106-109], cryptosystems [110,111], encryption [112,113], etc.

In the chaos literature, many different methodologies have been also proposed for the synchronization and anti-synchronization of chaotic systems such as PC method [114], active control [115-125], time-delayed feedback control [126,127], adaptive control [128-139], sampled-data feedback control [140-142], backstepping control [143-149], sliding mode control [150-158], etc.

The minimum dimension for an autonomous, continuous-time, hyperchaotic system is four. Since the discovery of the first hyperchaotic system by Rössler in 1979 [159], many 4-D hyperchaotic systems have been reported in the literature such as hyperchaotic Lorenz system [160], hyperchaotic Lü system [161], hyperchaotic Chen system [162], hyperchaotic Wang system [163], hyperchaotic Newton-Leipnik system [164], hyperchaotic Vaidyanathan systems [165-167], etc.

In this research work, a novel 4-D hyperchaotic Rikitake dynamo system has been proposed, which is derived by adding a feedback control to the famous 3-D Rikitake two-disk dynamo system (1958) found by Rikitake [168]. An extensive study by Cook and Roberts (1970) established chaos in the Rikitake two-disk dynamo system [169].

The 4-D novel hyperchaotic Rikitake dynamo system proposed in this research work has the Lyapunov exponents $L_1 = 0.09136, L_2 = 0.02198, L_3 = 0$ and $L_4 = -2.11190$.

Also, the Kaplan-Yorke dimension of the 4-D hyperchaotic Rikitake dynamo system is found as $D_{KY} = 3.05367$.

It is noted that the novel hyperchaotic Rikitake dynamo system does not have any equilibrium point, which is a novel feature of the system.

Some examples of hyperchaotic system without an equilibrium point are Wang's system [170], Pham's systems [171,172], etc.

According to a new classification of chaotic dynamics [173,174], there are two types of attractors: *self-excited* attractors and *hidden* attractors. A self-excited attractor has a basin of attraction that is excited from unstable equilibria. In contrast, hidden attractor cannot be found by using a numerical method in which a trajectory started from a point on the unstable manifold in the neighbourhood of an unstable equilibrium [173]. Studying hyperchaotic systems with hidden attractors is still an open research problem[175,176].

The paper is organized as follows. In Section 2, we describe the equations and phase portraits of the novel hyperchaotic Rikitake dynamo system. In Section 3, we derive the qualitative properties of the novel hyperchaotic Rikitake dynamo system. In Section 4, we derive an adaptive controller for the stabilization of 4-D novel hyperchaotic Rikitake dynamo system with unknown parameters.

In Section 5, we derive an adaptive controller for the synchronization of identical 4-D novel hyperchaotic Rikitake dynamo systems with unknown parameters. Section 6 contains a circuit realization of the proposed 4-D hyperchaotic Rikitake dynamo model. Section 7 concludes this research work with a summary of main results.

2. A 4-D Novel Hyperchaotic Rikitake Dynamo System

The frequent and irregular reversals of the Earth's magnetic field inspired a number of early studies involving electrical currents within the Earth's molten core. One of the first such models to report reversals was the Rikitake two-disk dynamo model [168].

The dynamics of the 3-D Rikitake two-disk dynamo system is described by

$$\begin{cases} \dfrac{dx_1}{dt} &= -ax_1 + x_2x_3 \\ \dfrac{dx_2}{dt} &= -ax_2 + x_1(x_3 - b) \\ \dfrac{dx_3}{dt} &= 1 - x_1x_2 \end{cases} \tag{1}$$

where x_1, x_2, x_3 are the states and a, b are constant, positive parameters. We note that the Rikitake dynamo system (1) has the same number of terms as the Lorenz chaotic system [2] but with an additional quadratic nonlinearity.

The parameter a represents resistive dissipation and the parameter b represents the difference in the angular velocities of the two disks.

The Rikitake dynamo system (1) depicts a chaotic attractor when the parameter values are taken as:

$$a = 1 \quad b = 1 \tag{2}$$

For numerical simulations, we take the initial conditions of the 3-D Rikitake two-disk dynamo system (1) as $x_1(0) = 0.8$, $x_2(0) = 0.2$, and $x_3(0) = 0.4$.

Also, for the parameter values given in (2), the Lyapunov exponents of the Rikitake dynamo system (1) are calculated as $L_1 = 0.12829, L_2 = 0$ and $L_3 = -2.12736$.

The 3-D portrait of the two-scroll chaotic attractor of the Rikitake dynamo system (1) for the parameter values given in (2) is depicted in Fig. 1.

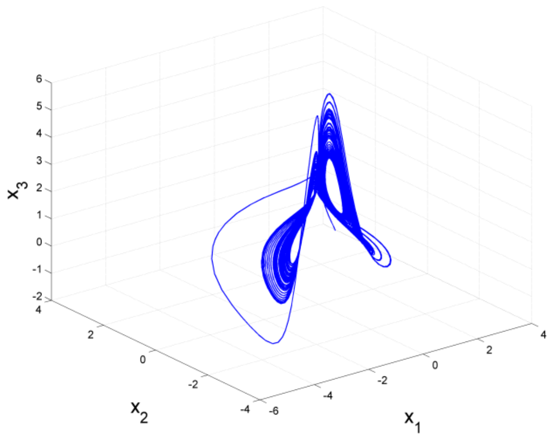

Fig. 1. The 2-scroll attractor of the Rikitake dynamo system in R^3.

In this work, we derive a novel 4-D hyperchaotic Rikitake dynamo model by adding a state feedback control to the 3-D Rikitake dynamo model (1) as follows.

$$\begin{cases} \frac{dx_1}{dt} &= -ax_1 + x_2x_3 - px_4 \\ \frac{dx_2}{dt} &= -ax_2 + x_1(x_3 - b) - px_4 \\ \frac{dx_3}{dt} &= 1 - x_1x_2 \\ \frac{dx_4}{dt} &= cx_2 \end{cases} \quad (3)$$

where x_1, x_2, x_3, x_4 are the state variables and a, b, c, p are constant, positive, parameters.

The 4-D system (3) is hyperchaotic when the parameter values are taken as

$$a = 1,\ b = 1,\ c = 0.7,\ p = 1.7 \quad (4)$$

For numerical simulations, we take the initial values of the 4-D system (3) as $x_1(0) = 0.8, x_2(0) = 0.2, x_3(0) = 0.4$ and $x_4(0) = 0.6$.

The Lyapunov exponents of the 4-D system (3) are calculated as:

$$\begin{cases} L_1 &= 0.09136 \\ L_2 &= 0.02198 \\ L_3 &= 0 \\ L_4 &= -2.11190 \end{cases} \quad (5)$$

Thus, the 4-D system (3) has two positive Lyapunov exponents, which shows that the system is hyperchaotic. Hence, the hyperchaotic Rikitake dynamo system (3) has very complex dynamics.

The Kaplan-Yorke dimension of the hyperchaotic Rikitake system (5) is obtained as:

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.05367 \quad (6)$$

Figures 2-5 depict the 3-D phase portraits of the 4-D hyperchaotic Rikitake dynamo system (3) in (x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_3, x_4), and (x_2, x_3, x_4) spaces, respectively.

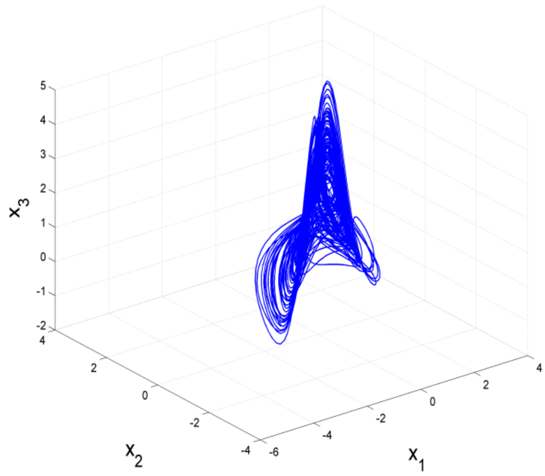

Fig. 2. The 3-D projection of the hyperchaotic Rikitake dynamo system on the (x_1, x_2, x_3) space.

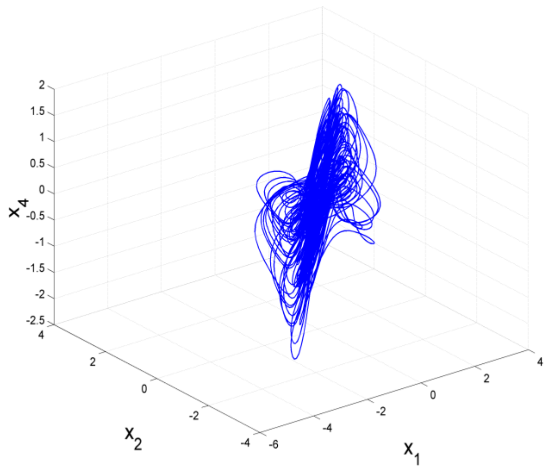

Fig. 3. The 3-D projection of the hyperchaotic Rikitake dynamo system on the (x_1, x_2, x_4) space.

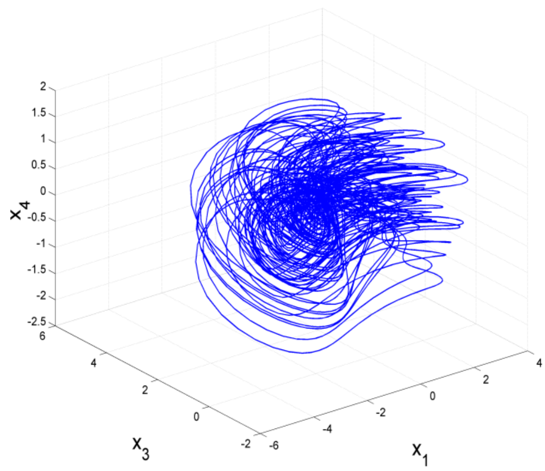

Fig. 4. The 3-D projection of the hyperchaotic Rikitake dynamo system on the (x_1, x_3, x_4) space.

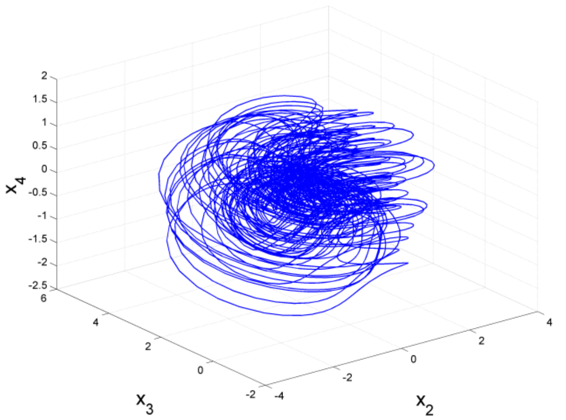

Fig. 5. The 3-D projection of the hyperchaotic Rikitake dynamo system on the (x_2, x_3, x_4) space.

3. Analysis of the 4-D Hyperchaotic Rikitake System

In this section, qualitative properties of the 4-D novel hyperchaotic system are detailed.

3.1. Dissipativity

In vector notation, we may express the system (3) as:

$$\frac{dx}{dt} = f(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2, x_3, x_4) \\ f_2(x_1, x_2, x_3, x_4) \\ f_3(x_1, x_2, x_3, x_4) \\ f_4(x_1, x_2, x_3, x_4) \end{bmatrix} \quad (7)$$

where

$$\begin{cases} f_1(x_1, x_2, x_3, x_4) &= -ax_1 + x_2 x_3 - px_4 \\ f_2(x_1, x_2, x_3, x_4) &= -ax_2 + x_1(x_3 - b) - px_4 \\ f_3(x_1, x_2, x_3, x_4) &= 1 - x_1 x_2 \\ f_4(x_1, x_2, x_3, x_4) &= cx_2 \end{cases} \quad (8)$$

We take the parameter values as in the chaotic case (4).

Let Ω be any region in $\boldsymbol{R}^4$ with a smooth boundary and also $\Omega(t) = \Phi_t(\Omega)$, where Φ_t is the flow of f.

Furthermore, let $V(t)$ denote the hypervolume of $\Omega(t)$.

By Liouville's theorem, we have

$$\frac{dV}{dt} = \int_{\Omega(t)} (\nabla \cdot f) dx_1 dx_2 dx_3 dx_4 \quad (9)$$

The divergence of the novel hyperchaotic Rikitake dynamo system (3) is easily found as:

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4} = -2a < 0 \quad (10)$$

Substituting (10) into (9), we obtain the first order ODE

$$\frac{dV}{dt} = \int_{\Omega(t)} (-a) dx_1 dx_2 dx_3 dx_4 = -aV \quad (11)$$

Integrating (11), we obtain the unique solution as:

$$V(t) = \exp(-at)\, V(0) \qquad \text{for all } t \geq 0 \quad (12)$$

From (12), we find that $V(t)$ shrinks to zero exponentially as $t \to \infty$.

Hence, the 4-D hyperchaotic Rikitake system (3) is dissipative and the asymptotic motion of the 4-D hyperchaotic system (3) settles exponentially onto a set of measure zero, *i.e.* a strange attractor.

3.2. Equilibrium Points

The equilibrium points of the novel hyperchaotic system (3) are obtained by solving the following system of equations with the parameter values as in the chaotic case (4):

$$\begin{cases} -ax_1 + x_2 x_3 - px_4 &= 0 \\ -ax_2 + x_1(x_3 - b) - px_4 &= 0 \\ 1 - x_1 x_2 &= 0 \\ cx_2 &= 0 \end{cases} \quad (13)$$

Since the last two equations of (13) contradict each other, there is no equilibrium point for the 4-D novel hyperchaotic Rikitake system (3).

3.3. Symmetry and Invariance

It is easy to check that the 4-D novel hyperchaotic Rikitake system (3) is invariant under the change of coordinates

$$(x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, x_3, -x_4) \quad (14)$$

Since the transformation (14) persists for all values of the system parameters, it follows that the 4-D hyperchaotic Rikitake system (3) has rotation symmetry about the x_3 axis. Hence, any non-trivial trajectory of the system (3) must have a twin trajectory.

It is also easy to check that the x_3 axis is invariant under the flow of the 4-D hyperchaotic Rikitake system (3). The invariant motion along the x_3 axis is characterized by the scalar dynamics $\dot{x}_3 = 1$, which is unstable.

3.4. Lyapunov Exponents and Kaplan-Yorke Dimension

For the chosen parameter values (4), the Lyapunov exponents of the novel 4-D system (3) are obtained using MATLAB as:

$$\begin{cases} L_1 &= 0.09136 \\ L_2 &= 0.02198 \\ L_3 &= 0 \\ L_4 &= -2.11190 \end{cases} \quad (15)$$

Since the spectrum of Lyapunov exponents (15) has two positive terms L_1, L_2, the system (3) is hyperchaotic.

We find that the sum of the Lyapunov exponents in (15) is negative, which shows that the hyperchaotic Rikitake dynamo system (3) is dissipative.

Also, the Kaplan-Yorke dimension of the novel hyperchaotic Rikitake dynamo system (3) is calculated as:

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.05367 \quad (16)$$

Fig. 5 depicts the dynamics of the Lyapunov exponents of the novel hyperchaotic Rikitake system (3).

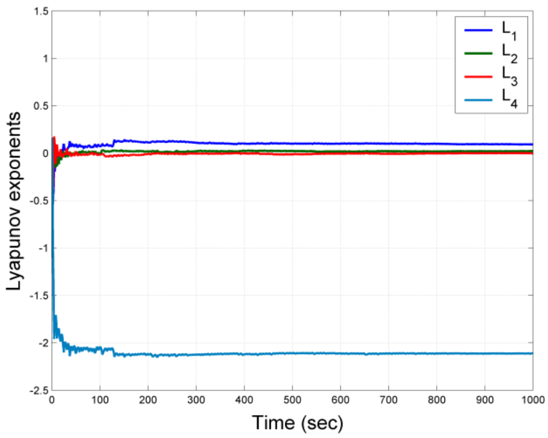

Fig. 6. The dynamics of the Lyapunov exponents of the hyperchaotic Rikitake dynamo system.

4. Adaptive Control of the 4-D Novel Hyperchaotic Rikitake Dynamo System

In this section, we construct an adaptive controller for globally stabilizing the 4-D novel hyperchaotic Rikitake dynamo system with unknown parameters.

We consider the controlled hyperchaotic system

$$\begin{cases} \frac{dx_1}{dt} &= -ax_1 + x_2x_3 - px_4 + u_1 \\ \frac{dx_2}{dt} &= -ax_2 + x_1(x_3 - b) - px_4 + u_2 \\ \frac{dx_3}{dt} &= 1 - x_1x_2 + u_3 \\ \frac{dx_4}{dt} &= cx_2 + u_4 \end{cases} \quad (17)$$

where x_1, x_2, x_3, x_4 are the state variables, a, b, c, p are unknown, constant, parameters and u_1, u_2, u_3, u_4 are adaptive controls to be designed using estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{p}(t)$ of the unknown parameters a, b, c, p, respectively.

We consider the adaptive controller defined by

$$\begin{cases} u_1 &= \hat{a}(t)x_1 - x_2x_3 + \hat{p}(t)x_4 - k_1x_1 \\ u_2 &= \hat{a}(t)x_2 - x_1\left(x_3 - \hat{b}(t)\right) + \hat{p}(t)x_4 - k_2x_2 \\ u_3 &= -1 + x_1x_2 - k_3x_3 \\ u_4 &= -\hat{c}(t)x_2 - k_4x_4 \end{cases} \quad (18)$$

where k_1, k_2, k_3, k_4 are positive gain constants, and $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{p}(t)$ are estimates of the unknown parameters a, b, c, p, respectively.

Substituting (18) into (17), we obtain

$$\begin{cases} \frac{dx_1}{dt} &= -(a - \hat{a}(t))x_1 - (p - \hat{p}(t))x_4 - k_1x_1 \\ \frac{dx_2}{dt} &= -(a - \hat{a}(t))x_2 - \left(b - \hat{b}(t)\right)x_1 \\ &\quad -(p - \hat{p}(t))x_4 - k_2x_2 \\ \frac{dx_3}{dt} &= -k_3x_3 \\ \frac{dx_4}{dt} &= (c - \hat{c}(t))x_2 - k_4x_4 \end{cases} \quad (19)$$

The parameter estimation errors are defined by

$$\begin{cases} e_a(t) &= a - \hat{a}(t) \\ e_b(t) &= b - \hat{b}(t) \\ e_c(t) &= c - \hat{c}(t) \\ e_p(t) &= p - \hat{p}(t) \end{cases} \quad (20)$$

Using (20), the closed-loop state dynamics (19) can be simplified as follows:

$$\begin{cases} \frac{dx_1}{dt} &= -e_ax_1 - e_px_4 - k_1x_1 \\ \frac{dx_2}{dt} &= -e_ax_2 - e_bx_1 - e_px_4 - k_2x_2 \\ \frac{dx_3}{dt} &= -k_3x_3 \\ \frac{dx_4}{dt} &= e_cx_2 - k_4x_4 \end{cases} \quad (21)$$

Differentiating (22) with respect to t, we get

$$\begin{cases} \frac{de_a}{dt} &= -\frac{d\hat{a}}{dt} \\ \frac{de_b}{dt} &= -\frac{d\hat{b}}{dt} \\ \frac{de_c}{dt} &= -\frac{d\hat{c}}{dt} \\ \frac{de_p}{dt} &= -\frac{d\hat{p}}{dt} \end{cases} \quad (22)$$

Next, we use Lyapunov stability theory for finding an update law for the parameter estimates.

Consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + e_a^2 + e_b^2 + e_c^2 + e_p^2\right), \quad (23)$$

which is positive definite on R^8.

Differentiating V along the trajectories of (21) and (22), we get

$$\begin{aligned} \frac{dV}{dt} = &-k_1x_1^2 - k_2x_2^2 - k_3x_3^2 - k_4x_4^2 \\ &+e_a\left[-x_1^2 - x_2^2 - \frac{d\hat{a}}{dt}\right] \\ &+e_b\left[-x_1x_2 - \frac{d\hat{b}}{dt}\right] + e_c\left[x_2x_4 - \frac{d\hat{c}}{dt}\right] \\ &+e_p\left[-x_4(x_1 + x_2) - \frac{d\hat{p}}{dt}\right] \end{aligned} \quad (24)$$

In view of (24), we take the parameter update law as

$$\begin{cases} \frac{d\hat{a}}{dt} &= -x_1^2 - x_2^2 \\ \frac{d\hat{b}}{dt} &= -x_1x_2 \\ \frac{d\hat{c}}{dt} &= x_2x_4 \\ \frac{d\hat{p}}{dt} &= -x_4(x_1 + x_2) \end{cases} \quad (25)$$

Next, we state and prove the main result of this section.

Theorem 1. *The novel hyperchaotic Rikitake dynamo system (17) is globally and exponentially stabilized by the adaptive control law (18) and the parameter update law (25) for all initial conditions, where k_1, k_2, k_3, k_4 are positive gain constants.*

Proof. We prove this result using Lyapunov stability theory.

For this purpose, we consider the quadratic Lyapunov function V defined by (23), which is positive definite on R^8.

Substituting the parameter update law (25) into (24), we obtain the time derivative of V as:

$$\frac{dV}{dt} = -k_1x_1^2 - k_2x_2^2 - k_3x_3^2 - k_4x_4^2, \tag{26}$$

which is a negative semi-definite function on R^8.

Thus, we can conclude that the state vector $x(t)$ and the parameter estimation error are globally bounded.

We define $k = \min\{k_1, k_2, k_3, k_4,\}$. Then we get

$$\frac{dV}{dt} \le -k\|x\|^2 \text{ or } k\|x\|^2 \le -\frac{dV}{dt} \tag{27}$$

Integrating the inequality (27) from 0 to t, we get

$$k\int_0^t \|x(\tau)\|^2 d\tau \le V(0) - V(t) \tag{28}$$

From (28), it follows that $x(t) \in L_2$. Using (21), we can conclude that $\dot{x} \in L_\infty$.

Thus, using Barbalat's lemma [177], we conclude that $x(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $x(0) \in R^4$.

This completes the proof. ■

For numerical simulations, the parameter values of the novel hyperchaotic Rikitake dynamo system (17) are taken as in the hyperchaotic case, *viz.* $a = 1$, $b = 1$, $c = 0.7$ and $p = 1.7$. We take the gain constants as $k_i = 5$ for $i = 1,2,3,4$.

The initial conditions of the hyperchaotic Rikitake dynamo system (17) are taken as $x_1(0) = 3.5, x_2(0) = 1.7$, $x_3(0) = -4.5$, and $x_4(0) = 2.8$.

The initial conditions of the parameter estimates are taken as $\hat{a}(0) = 7$, $\hat{b}(0) = 10$, $\hat{c}(0) = 22$ and $\hat{p}(0) = 18$. Figure 7 describes the time-history of the state $x(t)$.

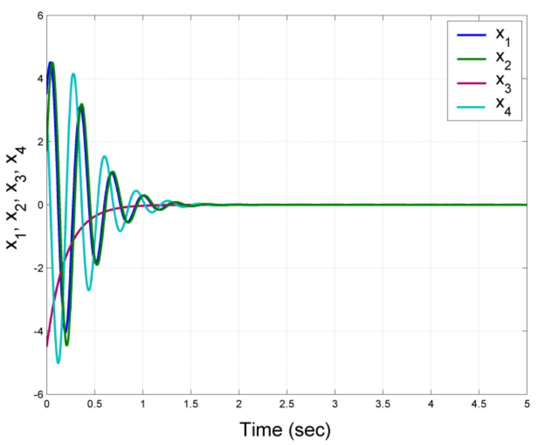

Fig. 7. Time-history of the controlled states x_1, x_2, x_3, x_4 of the hyperchaotic Rikitake dynamo system.

5. Adaptive Synchronization of Identical 4-D Novel Hyperchaotic Rikitake Dynamo Systems

In this section, we construct an adaptive synchronizer for global synchronization of identical 4-D novel hyperchaotic Rikitake dynamo systems. The adaptive synchronizer design is carried out using Lyapunov stability theory.

As the master system, we take the hyperchaotic Rikitake dynamo system

$$\begin{cases} \frac{dx_1}{dt} &= -ax_1 + x_2x_3 - px_4 \\ \frac{dx_2}{dt} &= -ax_2 + x_1(x_3 - b) - px_4 \\ \frac{dx_3}{dt} &= 1 - x_1x_2 \\ \frac{dx_4}{dt} &= cx_2 \end{cases} \tag{29}$$

where x_1, x_2, x_3, x_4 are state variables and a, b, c, p are unknown, constant, parameters.

As the slave system, we take the novel hyperchaotic Rikitake dynamo system

$$\begin{cases} \frac{dy_1}{dt} &= -ay_1 + y_2y_3 - py_4 + u_1 \\ \frac{dy_2}{dt} &= -ay_2 + y_1(y_3 - b) - py_4 + u_2 \\ \frac{dy_3}{dt} &= 1 - y_1y_2 + u_3 \\ \frac{dy_4}{dt} &= cy_2 + u_4 \end{cases} \tag{30}$$

where y_1, y_2, y_3, y_4 are state variables and u_1, u_2, u_3, u_4 are adaptive controls.

The complete synchronization error between the systems (29) and (30) is defined as:

$$\begin{cases} e_1 &= y_1 - x_1 \\ e_2 &= y_2 - x_2 \\ e_3 &= y_3 - x_3 \\ e_4 &= y_4 - x_4 \end{cases} \tag{31}$$

The error dynamics is easily obtained as:

$$\begin{cases} \frac{de_1}{dt} &= -ae_1 - pe_4 + y_2y_3 - x_2x_3 + u_1 \\ \frac{de_2}{dt} &= -ae_2 - pe_4 - be_1 + y_1y_3 - x_1x_3 + u_2 \\ \frac{de_3}{dt} &= -y_1y_2 + x_1x_2 + u_3 \\ \frac{de_4}{dt} &= ce_2 + u_4 \end{cases} \tag{32}$$

We consider the adaptive controller defined by

$$\begin{cases} u_1 &= \hat{a}(t)e_1 + \hat{p}(t)e_4 - y_2y_3 + x_2x_3 - k_1e_1 \\ u_2 &= \hat{a}(t)e_2 + \hat{p}(t)e_4 + \hat{b}(t)e_1 \\ & \quad -y_1y_3 + x_1x_3 - k_2e_2 \\ u_3 &= y_1y_2 - x_1x_2 - k_3e_3 \\ u_4 &= -\hat{c}(t)e_2 - k_4e_4 \end{cases} \tag{33}$$

where k_1, k_2, k_3, k_4 are positive gain constants, and $\hat{a}(t), \hat{b}(t)$, $\hat{c}(t), \hat{p}(t)$ are estimates of the unknown parameters a, b, c, p, respectively.

Substituting (35) into (34), we get

$$\begin{cases} \frac{de_1}{dt} &= -(a - \hat{a}(t))e_1 - (p - \hat{p}(t))e_4 - k_1e_1 \\ \frac{de_2}{dt} &= -(a - \hat{a}(t))e_2 - (p - \hat{p}(t))e_4 \\ & \quad -(b - \hat{b}(t))e_1 - k_2e_2 \\ \frac{de_3}{dt} &= -k_3e_3 \\ \frac{de_4}{dt} &= (c - \hat{c}(t))e_2 - k_4e_4 \end{cases} \tag{34}$$

The parameter estimation errors are defined by

$$\begin{cases} e_a(t) &= a - \hat{a}(t) \\ e_b(t) &= b - \hat{b}(t) \\ e_c(t) &= c - \hat{c}(t) \\ e_p(t) &= p - \hat{p}(t) \end{cases} \tag{35}$$

Substituting (35) into the error dynamics (34), we get

$$\begin{cases} \frac{de_1}{dt} &= -e_a e_1 - e_p e_4 - k_1 e_1 \\ \frac{de_2}{dt} &= -e_a e_2 - e_p e_4 - e_b e_1 - k_2 e_2 \\ \frac{de_3}{dt} &= -k_3 e_3 \\ \frac{de_4}{dt} &= e_c e_2 - k_4 e_4 \end{cases} \tag{36}$$

Differentiating (35) with respect to t, we get

$$\begin{cases} \frac{de_a}{dt} &= -\frac{d\hat{a}}{dt} \\ \frac{de_b}{dt} &= -\frac{d\hat{b}}{dt} \\ \frac{de_c}{dt} &= -\frac{d\hat{c}}{dt} \\ \frac{de_p}{dt} &= -\frac{d\hat{p}}{dt} \end{cases} \tag{37}$$

Consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_p^2\right), \tag{38}$$

which is positive definite on R^8.

Differentiating V along the trajectories of (36) and (37), we get

$$\begin{aligned} \frac{dV}{dt} = & -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \\ & + e_a\left[-e_1^2 - e_2^2 - \frac{d\hat{a}}{dt}\right] + e_b\left[-e_1 e_2 - \frac{d\hat{b}}{dt}\right] \\ & + e_c\left[e_2 e_4 - \frac{d\hat{c}}{dt}\right] + e_p\left[-(e_1 + e_2)e_4 - \frac{d\hat{p}}{dt}\right] \end{aligned} \tag{39}$$

In view of (39), we take the parameter update law as:

$$\begin{cases} \frac{d\hat{a}}{dt} &= -e_1^2 - e_2^2 \\ \frac{d\hat{b}}{dt} &= -e_1 e_2 \\ \frac{d\hat{c}}{dt} &= e_2 e_4 \\ \frac{d\hat{p}}{dt} &= -(e_1 + e_2)e_4 \end{cases} \tag{40}$$

Next, we shall establish the main result of this section, viz. the adaptive synchronization of the identical hyperchaotic Rikitake dynamo systems (29) and (30). We have used Lyapunov stability theory to establish this result.

Theorem 2. *The novel hyperchaotic Rikitake dynamo systems (29) and (30) are globally and exponentially synchronized by the adaptive control law (33) and the parameter update law (40) for all initial conditions, where k_1, k_2, k_3, k_4 are positive gain constants.*

Proof. We prove this result using Lyapunov stability theory.

For this purpose, we consider the quadratic Lyapunov function V defined by (40), which is positive definite on R^8.

Substituting the parameter update law (40) into (39), we obtain the time derivative of Vas:

$$\frac{dV}{dt} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \tag{41}$$

Since $\frac{dV}{dt}$ is a negative semi-definite function on R^8, we can conclude that the anti-synchronization vector $e(t)$ and the parameter estimation error are globally bounded.

We define $k = \min\{k_1, k_2, k_3, k_4\}$.

Then we get

$$\frac{dV}{dt} \le -k\|e\|^2 \quad \text{or} \quad k\|e\|^2 \le -\frac{dV}{dt} \tag{42}$$

Integrating the inequality (42) from 0 to t, we get

$$k\int_0^t \|e(\tau)\|^2 d\tau \le V(0) - V(t) \tag{43}$$

From (43), it follows that $e(t) \in L_2$. Using (36), we can conclude that $\dot{e} \in L_\infty$.

Thus, using Barbalat's lemma [177], we conclude that $e(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $e(0) \in R^4$. This completes the proof. ■

For numerical simulations, the parameter values of the novel hyperchaotic Rikitake dynamo systems are taken as in the hyperchaotic case, *viz.* $a = 1, b = 1, c = 0.7$ and $p = 1.7$.

We take the gain constants as $k_i = 5$ for $i = 1, 2, 3, 4$.

The initial conditions of the master system (29) are taken as $x_1(0) = 3.4, x_2(0) = 1.8,\ x_3(0) = 1.2$ and $x_4 = -1.6$.

The initial conditions of the slave system (30) are taken as $y_1(0) = 1.2, y_2(0) = 6.2,\ y_3(0) = -2.7$ and $y_4 = 4.5$.

The initial conditions of the parameter estimates are taken as $\hat{a}(0) = 6,\ \hat{b}(0) = 12,\ \hat{c}(0) = 21$ and $\hat{p}(0) = 14$.

Figures 8-11 describe the complete synchronization of the states of the novel hyperchaotic Rikitake dynamo systems (29) and (30).

Figure 12 describes the time-history of the synchronization error $e(t)$.

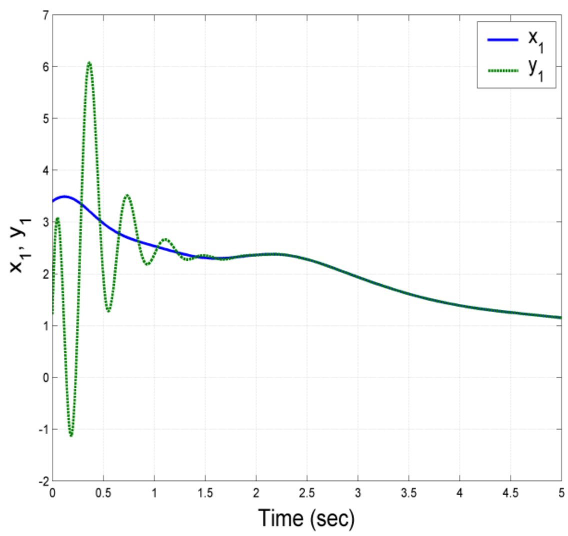

Fig. 8. The synchronization of the states x_1 and y_1.

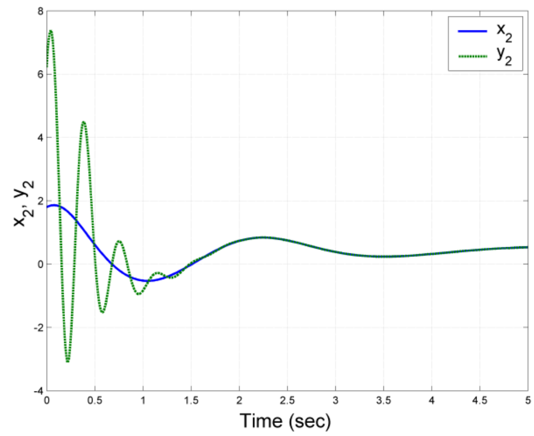

Fig. 9. The synchronization of the states x_2 and y_2.

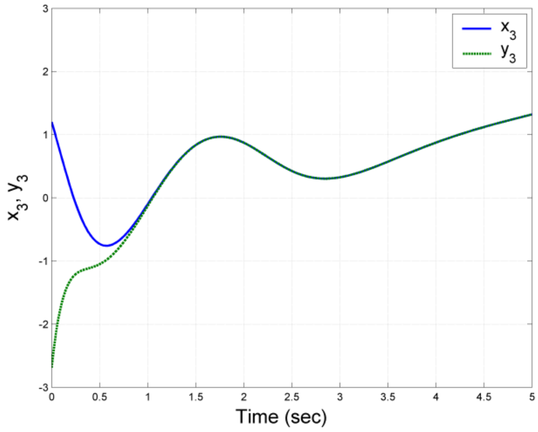

Fig. 10. The synchronization of the states x_3 and y_3.

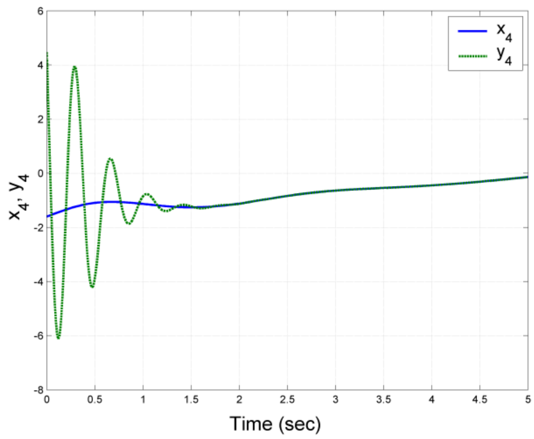

Fig. 11. The synchronization of the states x_4 and y_4.

6. Circuit Simulation

In this Section, an electronic circuit is constructed to implement hyperchaotic system without equilibrium (3). The circuit in Fig. 13 has been designed using a general approach based on operational amplifiers [30,35,36].

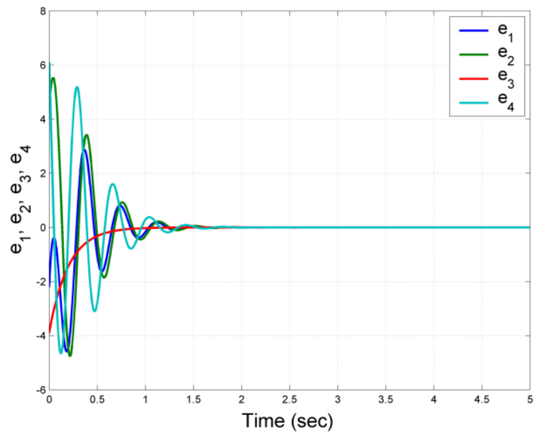

Fig. 12. Time-history of the synchronization error.

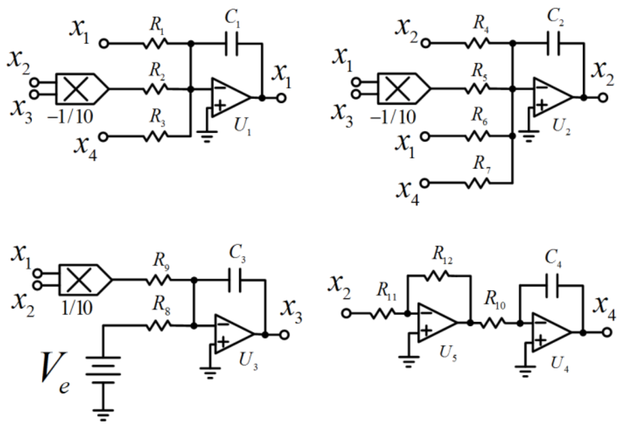

Fig. 13. Schematic of the circuit realizing the novel 4-D hyperchaotic Rikitake dynamo model (3).

By applying Kirchhoff's laws to the electronic circuit in Fig. 13, its nonlinear equations are given as

$$\begin{cases} \dfrac{dv_{C_1}}{dt} = -\dfrac{1}{R_1C_1}v_{C_1} + \dfrac{1}{10R_2C_1}v_{C_2}v_{C_3} - \dfrac{1}{R_3C_1}v_{C_4} \\ \dfrac{dv_{C_2}}{dt} = -\dfrac{1}{R_4C_2}v_{C_2} + \dfrac{1}{10R_5C_2}v_{C_1}v_{C_3} - \dfrac{1}{R_6C_2}v_{C_1} - \dfrac{1}{R_7C_2}v_{C_4} \\ \dfrac{dv_{C_3}}{dt} = \dfrac{1}{R_8C_3}V_e - \dfrac{1}{10R_9C_3}v_{C_1}v_{C_2} \\ \dfrac{dv_{C_4}}{dt} = \dfrac{R_{12}}{R_{10}R_{11}C_4}v_{C_2} \end{cases} \tag{44}$$

where v_{C_1}, v_{C_2}, v_{C_3}, v_{C_4} are the voltages across the capacitors C_1, C_2, C_3 and C_4, respectively. It is noting that the state variables x_1, x_2, x_3, x_4 of 4-D hyperchaotic Rikitake dynamo model (3) are the voltages v_{C_1}, v_{C_2}, v_{C_3}, v_{C_4} respectively.

The values of the electronic components in Fig. 13 are chosen to match parameters of 4-D hyperchaotic Rikitake dynamo model (3) as follows: $R_1 = R_4 = R_6 = R_8 = R_{11} = R_{12} =$ 10kΩ, $R_2 = R_5 = R_9 =$ 1kΩ, $R_3 = R_7 =$ 5.882kΩ, $R_{10} =$ 14.286kΩ, $C_1 = C_2 = C_3 = C_4 =$ 10nF, and $V_e = -1V_{DC}$. The power supplies of all active devices are ±15Volts.

The designed circuit is implemented by using the electronic simulation package Cadence OrCAD and obtained phase portraits are presented in Figs. 14-16.

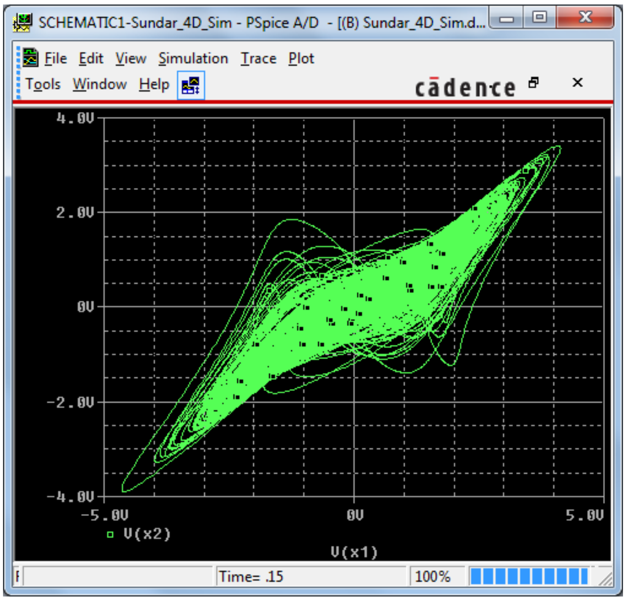

Fig. 14. Hyperchaotic attractor obtained from the proposed circuit in (v_{C_1}, v_{C_2})-plane

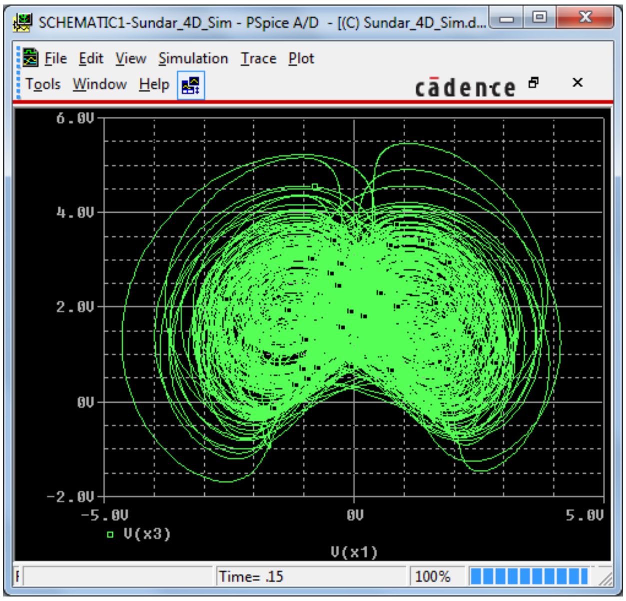

Fig. 15. Hyperchaotic attractor obtained from the proposed circuit in (v_{C_1}, v_{C_3})-plane

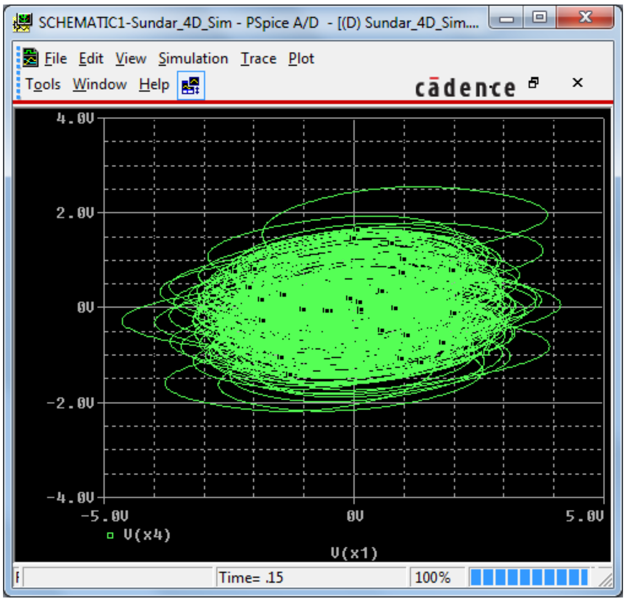

Fig. 16. Hyperchaotic attractor obtained from the proposed circuit in (v_{C_1}, v_{C_4})-plane

7. Conclusion

In this paper, a new 4-D Rikitake dynamo model has been investigated. It is interesting that such Rikitake dynamo model can exhibit hyperchaos although it possesses no equilibrium points. As the result, this system can be classified as a hyperchaotic system with hidden strange attractor. Qualitative properties of the 4-D novel hyperchaotic system are studied. In addition, an adaptive controller and an adaptive synchronizer for such system have been proposed. Moreover, an electronic circuit modeling this new hyperchaotic system has been designed using off-the-shelf components to confirm the feasibility of theoretical model.

Recently, discovering systems with hidden attractors has come a new attractive research topic because of their practical and theoretical importance. Especially there are only few works reporting hyperchaotic hidden strange attractor. Hence this Rikitake dynamo model should be further studied in future works.

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