Journal of Engineering Science and Technology Review 8 (2) (2015) 185 - 191
Special Issue on Synchronization and Control of Chaos: Theory, Methods and Applications

JOURNAL OF
Engineering Science and Technology Review

www.jestr.org

Research Article

Synchronization of Generalized Chua's Chaotic Oscillators in Small-world Topology

A. G. Soriano-Sánchez[1], C. Posadas-Castillo*[, 1] and M. A. Platas-Garza [1]

[1]*Universidad Autónoma de Nuevo León - Facultad de Ingeniería Mecánica y Eléctrica (UANL-FIME)*
Av. Pedro de Alba, S.N., Cd. Universitaria, C.P. 66451, San Nicolás de los Garza, N.L., México.

Received 2 September 2014; Revised 4 October 2014; Accepted 25 October 2014

Abstract

By using the generalized Chua's chaotic oscillator as fundamental unit, a complex network displaying the small-world property is synchronized. Authors resort to existing information about that long-range connection in a network increases the data flow from one unit to another to achieve synchronization under a small fixed coupling strength. Computer simulations are provided to show this chaos synchronization based on complex systems theory.

Keywords: Small-world, synchronization, complex networks, multi-scroll chaotic oscillators.

1. Introduction

Chaotic oscillators have attracted the interest of the researchers who have studied them extensively because of their potential application in different areas of science. It is of special interest the Chua's chaotic oscillator which has been generalized multiple times based on two categories: those in amending the nonlinear function and those that increase the system dimension [1]. The chaotic oscillator, which is used, has a nonlinear function composed by multi breaking points which allow the system generate multi-scroll attractors and is reported in [2-4] as *the generalized Chua's chaotic oscillator*.

Since L.M. Pecora and T.L. Carroll synchronized two chaotic oscillators in 1990 [5], chaos and its synchronization have received a great deal of attention. In the last 20 years, a considerable body of knowledge has been established on these topics, specially on synchronization. Many methods to synchronize coupled chaotic oscillators in a variety of topologies and configurations have been suggested [6-28]. When using control schemes, synchronization is regularly carried out in pairs of oscillators (master - slave) [6-19]; in the other hand, when coupled oscillators are seen as a community (network), synchronization is achieved through different techniques [20-28] that not only take into account the oscillators but also the topology, which plays a key role in this process.

Complex networks synchronization is a topic extensively studied nowadays, the most important results that have been derived from this research are, first the discovery that the behavior of many biological and non-biological systems can be modeled by the dynamics of complex networks: modeling of the human brain [29], modeling of society (individuals and relations existing between them) [30], the spread of epidemics in a population [31-34], modeling of influence of relationships in collaborative networks [35], modeling of economic systems [36]; second the effect of topology on the realization of system processes: the synchronization of pyloric central pattern generator of the lobster [20], the presence of Alzheimer in a human brain [37], stable growth of a neurons population, the spread of diseases in a population [38]; from a non-biological point of view: fast transmission of information between individuals [39, 40], modeling of trains network [41], the generation of memory capacity in a system [40], spreading of rumors [42] for instance.

The above quoted papers are some results of the investigation of complex networks arranged in a particular way, small-world topology. The small-world networks have their beginning in the 1960's when Stanley Milgram performed an experiment that led to the well-known concept of six degrees of separation [52]. Such networks became very popular after D.J. Watts and S.H. Strogatz published the algorithm to introduce the small-world property to a regular network and showed that the resulting network fulfilled the main characteristics: high clustering coefficient and short average path length [43]. According to [43] and [45], Milgram's experiment consisted of randomly distributed letters to people in Nebraska to be sent to Boston by people who might know the consignee. Milgram found that it had only taken an average of six steps for a letter to get from Nebraska to Boston. He concluded that six was the average number of acquaintances separating people in the entire world. Later we briefly describe the Watts & Strogatz

* E-mail address: cornelio.posadascs@uanl.edu.mx
ISSN: 1791-2377

algorithm, however, for readers interested in small-world topic please refer to [44-50] and references therein.

In this paper we focus specifically on the effect of the small-world topology on the synchronization of a complex network. It is graphically shown that the introduction of the small-world property in the topology reduces the synchronization time. Authors resort to existing information that suggests that long-range links in a network allow information to flow faster between individuals (data packets, epidemics, rumors, etc) [31, 33, 39, 40] and [42].

The paper is organized as follows: in Section II a brief review on complex dynamical networks and small-world topology is given. In Section III, the model of the generalized Chua's chaotic oscillator, which will be used as fundamental oscillator of the complex networks, is presented. In Section IV, synchronization of N-coupled multi-scroll chaotic oscillators is exposed; the corresponding simulation results are provided also in this section. Finally, some conclusions are given in Section V.

2. Complex Networks

A complex network is defined as an interconnected set of oscillators (two or more), where each oscillator is a fundamental unit, with its dynamic depending of the nature of the network. Each oscillator is defined as follows

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i)+\mathbf{u}_i, \qquad \mathbf{x}_i(0), \qquad i=1,2,...,N \tag{1}$$

where $\boldsymbol{x}_i = [x_{i1} \quad x_{i2} \quad ... \quad x_{in}] \in \Re^n$ are the state variables of the oscillator i, $\mathbf{x}_i(0) \in \Re^n$ are the initial conditions and $\mathbf{u}_i \in \Re^n$ establishes the synchronization between two or more oscillators and is defined in [51] as follows:

$$\mathbf{u}_i = c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \qquad i=1,2,...,N \tag{2}$$

the constant c positive definite represents the coupling strength and $\boldsymbol{\Gamma} \in \Re^{n\times n}$ is a constant matrix linking coupled state variables. In this matrix, two nodes are linked through their k-th state variable. Assume that $\Gamma = \mathrm{diag}(r_1, r_2, \dots, r_n)$ is a diagonal matrix with $r_k = 1$ for a particular k and $r_j = 0$ for $j \neq k$.

The matrix $\mathbf{A} \in \Re^{N\times N}$ with elements a_{ij} is the coupling matrix which shows the connections between oscillators, if the oscillator i-th is connected to the oscillator j-th, then $a_{ij} = 1$, otherwise $a_{ij} = 0$ for $i \neq j$. The diagonal elements of $\mathbf{A}$ are defined as:

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij} = -\sum_{j=1, j\neq i}^{N} a_{ji}, \qquad i=1,2,...,N \tag{3}$$

The dynamical complex network (1) and (2) is said to achieve synchronization if

$$\mathbf{x}_1(t)=\mathbf{x}_2(t)=\dots=\mathbf{x}_N(t), \qquad t\to\infty \tag{4}$$

In this paper, a network with N identical generalized Chua's chaotic oscillator displaying small-world topology is considered.

2.1. Watts and Strogatz Small-world Model

In 1998 D.J. Watts and S.H. Strogatz proposed an algorithm to generate "small-world" networks that starts from a network with regular topology, in this case the nearest neighbor, which is a ring lattice with periodic boundary conditions [44]; the Watts-Strogatz model is created rewiring a small fraction of the links to new randomly chosen positions; the only restrictions are that no oscillator can have multiple links with another oscillator and no oscillator must have links with itself. In Fig. 1 an example of a small-world network created with this model is shown. Here, N is the size of the network, k is the periodic boundary condition of the nearest neighbor topology, node i is connected with $i \pm 1, i \pm 2, \dots, i \pm k$; p is the rewiring probability and Nkp are the number of long-range link.

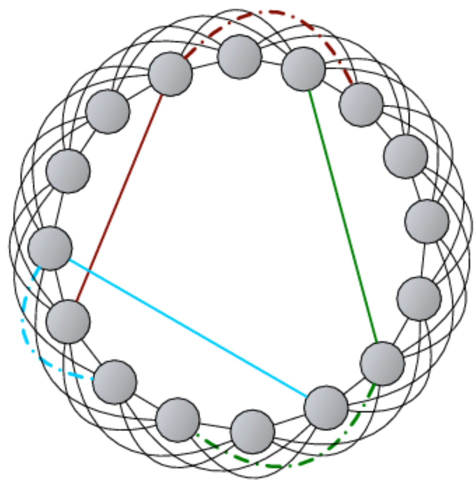

Fig. 1. Small-world network created rewiring some links to new randomly chosen positions. The dash-dot line was the link that we rewired; the solid line is the new position of the link chosen randomly.

Two of the most important features of complex networks are: first, *the clustering coefficient C*, which is the average fraction of pairs of neighbors of a oscillator that are also neighbors of each other, the clustering coefficient C_i of the oscillator i is defined as the ratio between the number E_i of edges that exist between k_i oscillators and the total number $k_i(k_i - 1)/2$, so

$$C_i = \frac{2E_i}{k_i(k_i-1)} \tag{5}$$

The clustering coefficient C of the whole network is the average of C_i over all i; second, *the average path length L*, which is defined as the distance between two oscillators averaged over all pairs of oscillators [53]. Due to the existence of long-range links, the small-world network has high clustering coefficient $C(p)$, and a short average path length $L(p)$.

As seen in Fig. 1, the long-range links allow us now reach any remote node faster; that is the main idea behind this research. Based on the above findings [31,33,39,40,42], we sought to demonstrate that introducing the small-world property will allow us to synchronize a complex network by modifying its topology through varying the probability while using a small coupling strength. Also we show graphically how the synchronization time decreases as the probability increases. For further details on small-world networks, go to [44-53] and references therein.

3. Generalized Chua's Chaotic Oscillator

In this section the generalized Chua's chaotic oscillator will be presented and briefly described.

The generalized Chua's chaotic oscillator is described by the following set of equations

$$\begin{cases} \dot{x} = \alpha\left[y - h(x)\right] \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y \end{cases} \tag{6}$$

and the nonlinearity

$$h(x) = m_{2q-1}x + \frac{1}{2}\sum_{l=1}^{2q-1}\left(m_{l-1} - m_l\right)\left(\left|x + b_l\right| - \left|x - b_l\right|\right) \tag{7}$$

The oscillator (6) exhibits a chaotic dynamic for parameters $\alpha = 9$ and $\beta = 100/7$ [4]. It can be seen that equation (7) is composed by multiple breaking points where q denotes a natural number. The generalized Chua's chaotic oscillator is described by three scalars and two parameter vectors $\{\alpha, \beta, q, m, b\}$, where $m = [m_0\ m_1\ \ldots\ m_{2q-1}]$ and $b = [b_1\ b_2\ \ldots\ b_{2q-1}]$. Various values for parameters and parameter vectors which can generate different amount of scrolls can be found in [4]. Fig. 2 shows an example of a chaotic attractor that can be obtained with the set of equations (6).

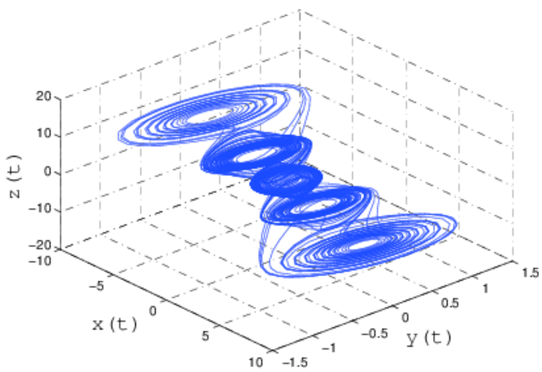

Fig. 2. View x−y−z of a 5-scrolls chaotic attractor of the generalized Chua's chaotic oscillator.

The equilibrium points of the system (6) are given by

$$\begin{cases} h(x) = 0 \\ x = -z \\ y = 0 \end{cases} \tag{8}$$

The origin is an equilibrium point of the system (6); the equilibrium points left are generated by the condition $h(x) = 0$ and they are located at the intersection of the nonlinear function and the x-axis. These equilibrium points appear in pairs, they are denoted as $EQ_j = [x_{EQj}^{\pm}, 0, -x_{EQj}^{\pm}]$ and an example is shown in Fig. 3.

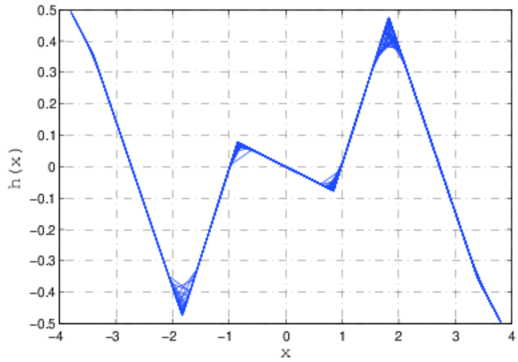

Fig. 3. Location of the equilibrium points generated by condition $h(x) = 0$, $EQ = [2.7, 1, 0, -1, -2.7]$.

4. Synchronization of *N*-coupled Generalized Chua's Chaotic Oscillators Arranged in Small-world Topology

In this section a small-world network of identical generalized Chua's chaotic oscillators will be synchronized. Authors are about to show that when existing long-range links in a complex network, synchronization is achievable easier even with a small coupling strength c. We will consider two chaotic oscillators as synchronized when the synchronization equation (9) is satisfied.

$$\lim_{t\to\infty}\left\|\left[x_1 \quad y_1 \quad z_1\right]^T - \left[x_2 \quad y_2 \quad z_2\right]^T\right\| = 0 \tag{9}$$

As mentioned above, the small-world property is introduced by using the Watts-Strogatz algorithm. The network to be synchronized has the following characteristics: network's size $N = 50$ and the periodic boundary condition $k = 5$; the rewiring probability p will be increased from zero until we have introduced the Nkp long-range links needed to achieve synchronization. From reference [53] we know that locally coupled networks, like the nearest-neighbor network, are difficult to coordinate, therefore, any dynamic process such as synchronization, it is very difficult to obtain, however, we expect to achieve synchronization by introducing some randomness in the network's topology.

4.1. Conditions for Synchronization by Using the Coupling Matrix

Suppose that there are no isolated clusters in the network, then the coupling matrix $\boldsymbol{A}$, obtained as explained in Section II, is a symmetric irreducible matrix, so one eigenvalue of $\mathbf{A}$ is zero and all the other eigenvalues are strictly negative, this means, $\lambda_{2,\ldots,N}(\mathbf{A}) < 0$.

Theorem 1 ([53]): *Consider the dynamical network (1). Let*

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N \tag{10}$$

be the eigenvalues of its coupling matrix $\mathbf{A}$. *Suppose that there exist an* $n \times n$ *diagonal matrix* $\mathbf{D} > 0$ *and two constants* $\bar{d} < 0$ *and* $\tau > 0$, *such that*

$$[Df(s(t))+d\Gamma]^T \mathbf{D}+\mathbf{D}[Df(s(t))+d\Gamma]\le -\tau \mathbf{I}_n \quad (11)$$

for all $d \le d\Gamma$ *here* $\mathbf{I}_n \in \Re^{n\times n}$ *is an unit matrix. If moreover,*

$$c\lambda_2 \le \bar{d} \quad (12)$$

then the synchronization state (4) is exponentially stable.

The coupling strength c, that determines the stability of the synchronization state (Eqn. (12)) and that is present in the control law (Eqn. (2)), is computed based on the following lemma:

Lemma 1 ([51]): *Consider network (1). Let* λ_2 *be the largest nonzero eigenvalue of the coupling matrix* $\mathbf{A}$ *of the network. The synchronization state of network (1) defined by* $\mathbf{x}_1(t)=\mathbf{x}_2(t)=\ldots=\mathbf{x}_N(t)$ *is asymptotically stable, if*

$$\lambda_2 \le -\frac{T}{c} \quad (13)$$

where $c > 0$ *is the coupling strength of the network and* $T > 0$ *is a positive constant such that zero is an exponentially stable point of the n-dimensional system*

$$\begin{aligned} \dot{z}_1 &= f_1(z) - Tz_1, \\ \dot{z}_2 &= f_2(z), \\ &\vdots \\ \dot{z}_n &= f_n(z). \end{aligned} \quad (14)$$

Note that system (14) is actually a single oscillator with self-feedback $-Tz_1$*. Condition (13) means that the entire network will synchronize provided that* λ_2 *is negative enough, e.g. it is sufficient to be less than* $-T/c$*, where* T *is a constant so that the self-feedback term* $-Tz_1$ *could stabilize an isolated oscillator.*

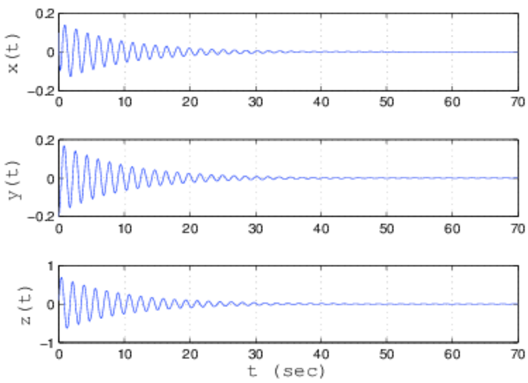

Fig. 4. A single generalized Chua's chaotic oscillator with self-feedback $-Tz_1$ where $T = 9$ is enough to stabilize all states of the isolated oscillator.

For the multi-scroll chaotic oscillator (6) the self-feedback term $-Tz_1$ with $T = 9$ stabilizes all states of the isolated oscillator, which is shown in Fig. 4. From Eqn. (13), we can determine that $c \ge -T/\lambda_2 \ge -9/\lambda_2$. Now, we have only to determine λ_2, which will vary as the network varies through the rewiring probability p. When $p = 0$ the complex network has the original nearest-neighbor topology, thus, we compute $\lambda_2 = -0.8484$ by using the network parameter given at the beginning of this section.

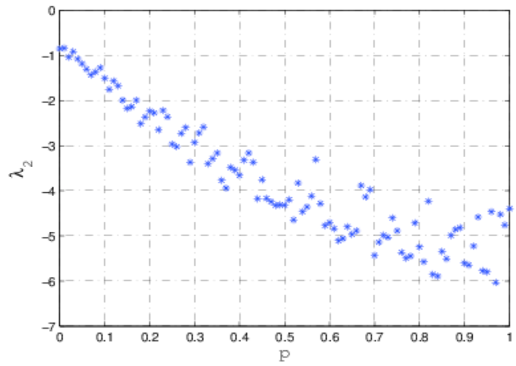

Fig. 5. Variation of λ_2 depending on the rewiring probability p of the complex network.

Fig. 5 shows the variation of the largest nonzero eigenvalue as a function of the rewiring probability p; as you can see, λ_2 has the smallest value when $p = 0$, therefore, the coupling strength lowest value is determined by

$$c \ge \left|\frac{9}{0.8484}\right| \ge 10.6082 \quad (15)$$

It was emphasized the calculation of c to show that it can be considerably reduced when information in the complex network flows more efficiently between nodes. Our interest is to show that synchronization can be achieved by varying the number of long-range links leaving fixed the value of the coupling strength in a small arbitrary value. In the remainder of the section synchronization results are presented.

4.2 Synchronization Results

The results we are about to present were obtained by synchronizing a network of $N = 50$ generalized Chua's chaotic oscillators, with a periodic boundary condition of $k = 5$ nearest-neighbor and varying the rewiring probability p from zero. The coupling strength was set at $c = 3$, which is the 28 percent of that obtained by the Wang & Chen theorem.

Due to the size of the network, the figure illustrating the topology and the coupling matrix are omitted, however, the latter can be calculated using the theory explained in Section 2.

Each oscillator of the small-world complex network is defined as follows:

$$\begin{cases} \dot{x}_i = \alpha[y_i + h(x_i)] + u_{i1} \\ \dot{y}_i = x_i - y_i + z_i, \qquad i = 1,\ldots,50 \\ \dot{z}_i = -\beta y_i \end{cases} \quad (16)$$

where

$$h(x_i) = m_{2q-1}x_i + \frac{1}{2}\sum_{l=1}^{2q-1}(m_{l-1}-m_l)\,(|x_i+b_l| - |x_i-b_l|) \quad (17)$$

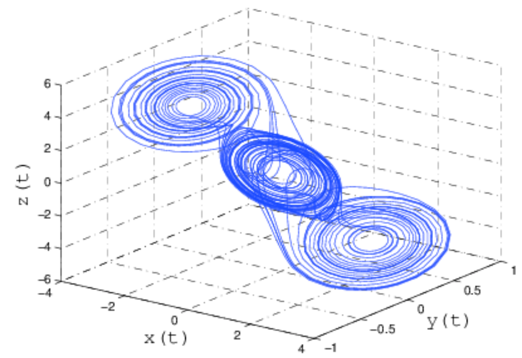

Fig. 6. View x–y–z of 3 scrolls chaotic attractor of the first generalized Chua's chaotic oscillator of the complex network.

Initial conditions were randomly generated for each oscillator without repeating them.

Fig. 6 shows the multi-scroll chaotic attractor generated by the final dynamic of the complex network composed by generalized Chua's chaotic oscillators.

Fig. 7 shows the three variables of each oscillator synchronizing; on the left, it can be seen all the variables of each oscillator starting from different initial conditions, as time passes the synchronization of the states is observed.

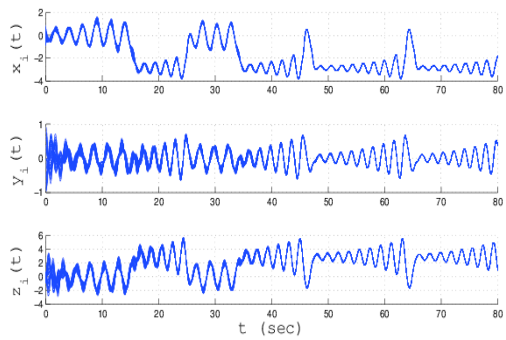

Fig. 7. Evolution of the three states $x_i(t)$, $y_i(t)$ and $z_i(t)$ for $i = 1, ..., 50$ by using $c = 3$.

To verify synchronization graphically, the phase portrait between some of the variables is shown and it is emphasized one of them; in Fig. 8 the phase portraits between x_i *vs.* x_j of nodes i = 1, 18, 32; j = 6, 38, 42 chosen randomly are shown; synchronization of the first state of some chaotic oscillators can be confirmed.

As can be seen, the small-world network synchronizes with a relatively small coupling strength (a third of the value obtained), however, the effect of interest to the authors is to know how the *Nkp* long-range links affect synchronization. Results are as follows: first, the error between all pairs of oscillators of the network were obtained, then all errors were averaged as denotes the following equation

$$e(t) = \frac{1}{\sum_{l=1}^{N-1}(N-l)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(\mathbf{x}_i(t) - \mathbf{x}_j(t) \right) \tag{18}$$

the result for the $x(t)$ and $y(t)$ states are shown in Fig. 9 and Fig. 10 respectively.

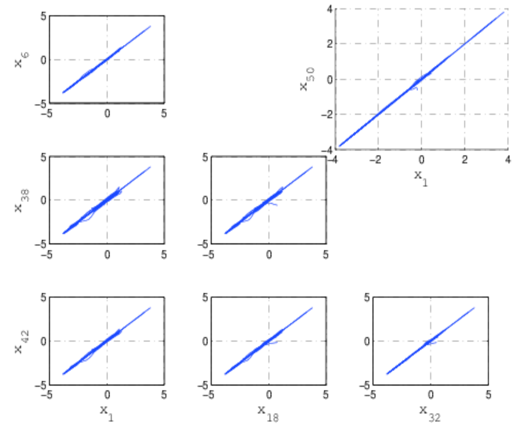

Fig. 8. Confirmed synchronization of the first state x_i *vs.* x_j i = 1, 18, 32; j = 6, 38, 42.

The four errors obtained for different values of p are centered around zero originally; however, for ease of viewing we applied an offset. The most important results in Figs. 9 & 10 are as follows: first, the average error of both $x(t)$ and $y(t)$ states converged to zero, this means that the synchronization condition (9) holds for any pair of oscillators. Second, the synchronization time is greatly reduced and can be clearly seen by comparing in Fig. 9 and Fig. 10 the corresponding error for p = 0.18 and p = 0.3.

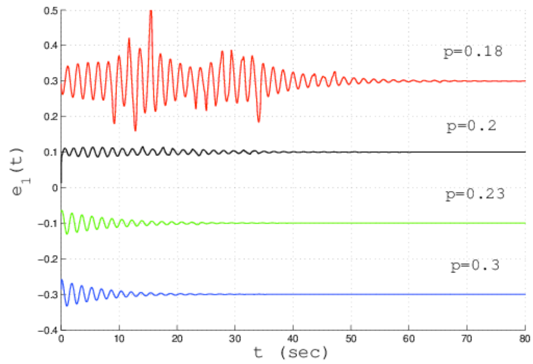

Fig. 9. Averaged error of $x(t)$ for all pairs of chaotic oscillator in the small-world network for different values of the rewiring probability p.

It is noteworthy that for values $0 < p < 0.18$ the network fails to synchronize, this is consistent with the above mentioned that states that locally clustered networks are difficult to coordinate. Furthermore, when it has reached p = 0.18 the network synchronizes and for larger values of $p > 0.18$ the synchronization time can be reduced without increasing the coupling strength by only varying the rewiring probability and therefore, the number of long-range links *Nkp*. Fig. 11 shows the synchronization time reduction for this particular case.

We have shown in Fig. 8 that two randomly chosen chaotic oscillators are synchronized; this is confirmed by the averaged error shown in Figs. 9 & 10 corresponding to the states $x(t)$ and $y(t)$ respectively. We have shown only the error of two states of all pairs of oscillators to summarize the information provided, but the $z(t)$ state also synchronizes and the error has a very similar behavior in time.

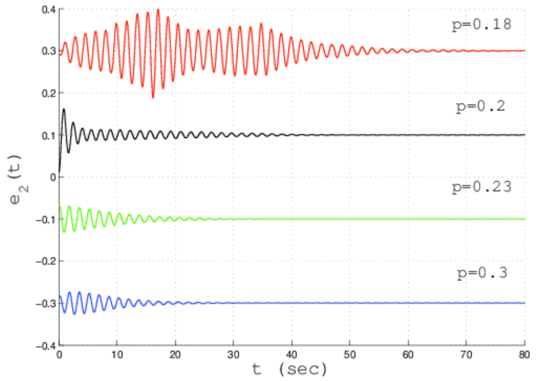

Fig. 10. Averaged error of $y(t)$ for all pairs of chaotic oscillator in the small-world network for different values of the rewiring probability p.

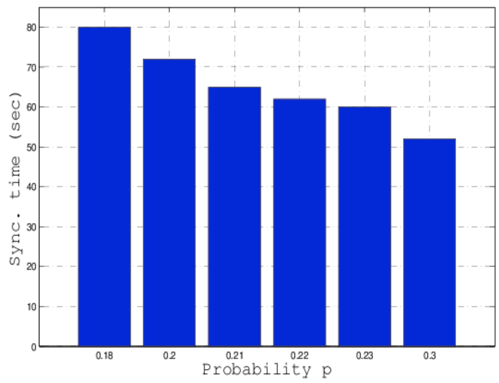

Fig. 11. Synchronization time reduction in the small-world network for different values of the rewiring probability p.

5. Conclusion

In this paper the synchronization of a complex network displaying the small-world topology was achieved. It was used a coupling signal in only one state of each chaotic oscillator with a fixed coupling strength c; the synchronization was achieved by varying the rewiring probability p as explained in the Watts-Strogatz model. The convergence to synchrony on some state of randomly chosen chaotic oscillators can be observed in the phase portraits or through the averaged error. There were synchronized complex networks composed by the generalized Chua's chaotic oscillator in small-world topology by modifying the network's topology. This procedure has proven to be effective to achieve synchronization, and without hesitation, it is the main result of this paper.

Acknowledgment

This work was supported by CONACYT México under Research Grant No. 166654, PAICYT under Research Grant IT956-11 and "Facultad de Ingeniería Mecánica y Eléctrica" (FIME-UANL).

References

1. M. Götz, U. Feldmann, and W. Schwarz, Synthesis of higher dimensional Chua circuits, IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, vol. 40(11) (1993).
2. J.A.K. Suykens, A. Huang, and L.O. Chua, A family of n-scroll attractors from a generalized Chua's circuit, Archiv fur Elektronik und Ubertragungstechnik, vol. 51(3), pp. 131-138 (1997).
3. J.A.K. Suykens and J. Vandewalle, Generation of n-double scrolls (n = 1, 2, 3, 4, ...), IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, vol. 40(11), (1993).
4. M.E. Yalçin, J.A.K. Suykens, and J.P.L. Vandewalle, Cellular neural networks, multi-scroll chaos and synchronization, World Scientific, Serie A vol. 50 (2005).
5. L.M. Pecora and T.L. Carroll, Synchronization in chaotic systems, Phys. Rev. Lett, vol. 64, pp. 821-824 (1990).
6. D. Ghosh and S. Banerjee, Projective synchronization of time-varying delayed neural network with adaptive scaling factors, Chaos, Solitons & Fractals, vol. 53, pp. 1-9 (2013).
7. H.K. Lam. Output-feedback synchronization of chaotic systems based on sum-of-squares approach, Chaos, Solitons & Fractals, vol. 41, pp. 262-2629 (2009).
8. D. López-Mancilla and C. Cruz-Hernández, Output synchronization of chaotic systems under non-vanishing perturbations, Chaos, Solitons & Fractals, vol. 37, pp. 1172-1186 (2008).
9. J.G. Barajas, K.P. Ramírez, C. Galarza, and R. Femat, Generalized synchronization of strictly different systems: Partial-state synchrony,. Chaos, Solitons & Fractals, vol. 45, pp. 193-204 (2012).
10. M. Sun, L. Tian, and Q. Jia, Adaptive control and synchronization of a four-dimensional energy resources system with unknown parameters, Chaos, Solitons & Fractals, vol. 39, pp. 1943-1949 (2009).
11. T.-L. Liao and N.-S. Huang, An observer-based approach for chaotic synchronization with applications to secure communications, IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, vol. 46(9) (1999).
12. Y. Feng, X. Yu, and L. Sun, Synchronization of uncertain chaotic systems using a single transmission channel, Chaos, Solitons & Fractals, vol. 35, pp. 755-762 (2008).
13. C. Cruz-Hernández, R.M. López-Gutiérrez, A.Y. Aguilar-Bustos, and C. Posadas-Castillo, Communicating encrypted information based on synchronized hyperchaotic maps, International Journal of Nonlinear Sciences & Numerical Simulation, vol. 11(5), pp. 337-349 (2010).
14. N. Chopra and M.W. Spong, Output synchronization of nonlinear systems with time delay in communication, In Proc. of the 45th IEEE Conference on Decision & Control, San Diego, CA, USA, December 13-15, (2006).
15. Z. Yan and P. Yu, Linear feedback control, adaptive feedback control and their combination for chaos (lag) synchronization of LC chaotic systems, Chaos, Solitons & Fractals, vol. 33, pp. 419-435 (2007).
16. M. Zribi, N. Smaoui, and Ha. Salim, Synchronization of the unified chaotic systems using a sliding mode controller, Chaos, Solitons & Fractals, vol. 42, pp. 3197-3209 (2009).
17. S.K. Agrawal, M. Srivastava, and S. Das, Synchronization of fractional order chaotic systems using active control method, Chaos, Solitons & Fractals, vol. 45, pp. 737-752 (2012).
18. A.C.J. Luo and F. Min, Synchronization dynamics of two different dynamical systems, Chaos, Solitons & Fractals, vol. 44, pp. 362-380 (2011).
19. L. Gámes-Guzmán, C. Cruz-Hernández, R.M. López-Gutiérrez, and E.E. García-Guerrero, Synchronization of Chua's circuits

with multi-scroll attractors: Application to communication, Commun. Nonlinear Sci. Numer. Simulat., vol. 14, pp. 2765-2775 (2009).
20. O. Cornejo-Pérez, G.C. Solís-Perales, and J.A. Arenas-Prado, Synchronization dynamics in a small pacemaker neuronal ensemble via a robust adaptive controller, Chaos, Solitons & Fractals, vol. 45, pp. 861-868 (2012).
21. N. Chopra, Output synchronization on strongly connected graphs, IEEE Transactions on Automatic Control, vol. 57(11) (2012).
22. H.F. Grip, T. Yang, A. Saberi, and A.A. Stoorvogel, Output synchronization for heterogeneous networks of non-introspective agents, Automatica, vol. 48, pp. 2444-2453 (2012).
23. Z. Li and G. Chen, Robust adaptive synchronization of uncertain dynamical networks, Physics Letters A, vol. 324, pp. 166-178 (2004).
24. Q. Song, J. Cao, and F. Liu, Synchronization of complex dynamical networks with nonidentical nodes, Physics Letters A, vol. 374, pp. 544-551 (2010).
25. J.H. Park, Synchronization of cellular neural networks of neutral type via dynamic feedback controller, Chaos, Solitons & Fractals, vol. 42, pp. 1299-1304 (2009).
26. C. Posadas-Castillo, E. Garza-González, D.A. Diaz-Romero, E. Alcorta-García, and C. Cruz-Hernández, Synchronization of irregular complex networks with chaotic oscillators: Hamiltonian systems approach, Journal of Applied Research and Technology, vol. 12 (2014).
27. A. Soriano-Sánchez, C. Posadas-Castillo, M.A. Platas-Garza, R.M. López-Gutiérrez, and C. Cruz-Hernández, Chaotic synchronization of irregular complex networks with multi-scroll attractors Genesio & Tesi 3-D, In Proc. of the 2013 International Conference on Systems, Control and Informatics (SCI 2013), ISBN: 978-1-61804-206-4Venice, Italy September (2013).
28. S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Complex networks: Structure and dynamics, Physics Reports, vol. 424, pp. 175-308 (2006).
29. O. Sporns, The human connectome: a complex network, Ann. N.Y. Acad. Sci., vol. 1224, pp. 109-125 (2011).
30. I. Kanovsky, Small world models for social network algorithms testing, Procedia Computer Science, vol. 1, pp. 2341-2344 (2012).
31. F.C. Santos, J.F. Rodrigues, and J.M. Pacheco, Epidemic spreading and cooperation dynamics on homogeneous small-world networks, Physical Review E, vol. 72, p. 056128 (2005).
32. A. Ramani, B. Grammaticos, and J. Satsuma, Modelling the dynamics of nonendemic epidemics, Chaos, Solitons &Fractals, vol. 40, pp. 491-496 (2009).
33. Z. Wang, H. Zhang, and Z. Wang, Multiple effects of self-protection on the spreading of epidemics, Chaos, Solitons & Fractals, vol. 61, pp. 1-7 (2014).
34. Z.J. Bao, Q.Y. Jiang, W.J. Yan, and Y.J. Cao, Stability of the spreading in small-world network with predictive controller, Physics Letters A, vol. 374, pp. 1560-1564 (2010).
35. P.V. Singh, The Small-world effect: The influence of macro-level properties of developer collaboration networks on open-source project success, ACM Transactions on Software Engineering and Methodology, vol. (20)2 (2010).
36. V. Latora and M. Marchiori, Economic small-world behavior in weighted networks, Eur. Phys. J. B, vol. 32, pp. 249-263 (2003).
37. C.J. Stam, B.F. Jones, G. Nolte, M. Breakspear, and Ph. Scheltens, Small-world networks and functional connectivity in Alzheimer's disease, Cerebral Cortex, vol. 17, pp. 92-99 (2007).
38. R.T. Gray, C.K.C. Fung, and P.A. Robinson, Stability of small-world networks of neural populations, Neurocomputing, vol. 72, pp. 1565-1574 (2009).
39. L. Guzmán-Vargas and R. Hernández-Pérez, Small-world topology and memory effects on decision time in opinion dynamics, Physica A, vol. 372, pp. 326-332 (2006).
40. N. Davey, B. Christianson, and R. Adam, High capacity associative memories and small world networks, In Proc. of IEEE International Joint Conference on Neural Networks (2004).
41. K.A. Seaton and L.M. Hackett, Stations, trains and small-world networks, Physica A, vol. 339, pp. 635-644 (2004).
42. D.H. Zanette, Dynamics of rumor propagation on small-world networks, Physical Review E, vol. 65, p. 041908 (2002).
43. D.J. Watts and S.H. Strogatz, Collective dynamics of "small-world" networks, Nature, vol. 393, pp. 440-442 (1998).
44. M.E.J. Newman, Models of the small world, Journal of Statistical Physics, vol. 101 (2000).
45. S. Milgram, The small-world problem, Psychology Today, vol. 1(1), pp. 61-67 (1967).
46. L.A.N. Amaral, A. Scala, M. Barthélémy, and H.E. Stanley, Classes of behavior of small-world networks, arXiv:cond-mat/0001458v1 [cond-mat.stat-mech] (2000).
47. X.F. Wang, G. Chen, Complex Networks: Small-World, Scale-Free and Beyond, IEEE Circuits and Systems Magazine, (2003).
48. R.V. Kulkarni, E. Almaas, D. Stroud, Exact results and scaling properties of small-world networks, arXiv:cond-mat/9908216v2 [cond-mat.stat-mech] (1999).
49. A. Barrat, and M. Weigt, On the properties of small-world network models, Eur. Phys. J. B, vol. 13, pp. 547-560 (2000).
50. L.A.N. Amaral, A. Scala, M. Barthélémy, and H.E. Stanley, Classes of small-world networks, In. Proc. of the National Academy of Sciences vol. 97(21), pp. 11149-11152 (2000).
51. X.F. Wang and G. Chen, Synchronization in small-world dynamical networks, International Journal of Bifurcation and Chaos, vol. 12(1), pp. 187-192 (2002).
52. J. Guare, Six degrees of separation: A play, Vintage, New York (1990).
53. X.F. Wang, Complex networks: Topology, dynamic and synchronization, International Journal of Bifurcation and Chaos, vol. 12(5), pp. 885-916 (2002).