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Complexity of Economical Systems

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Abstract

In this study new theoretical concepts are described concerning the interpretation of economical complex dynamics. In addition a summary of an extended algorithm of nonlinear time series analysis is provided which is applied not only in economical time series but also in other physical complex systems (e.g. [22, 24]). In general, Economy is a vast and complicated set of arrangements and actions wherein agents—consumers, firms, banks, investors, government agencies—buy and sell, speculate, trade, oversee, bring products into being, offer services, invest in companies, strategize, explore, forecast, compete, learn, innovate, and adapt. As a result the economic and financial variables such as foreign exchange rates, gross domestic product, interest rates, production, stock market prices and unemployment exhibit large-amplitude and aperiodic fluctuations evident in complex systems. Thus, the Economics can be considered as spatially distributed non-equilibrium complex system, for which new theoretical concepts, such as Tsallis non extensive statistical mechanics and strange dynamics, percolation, non-Gaussian, multifractal and multiscale dynamics related to fractional Langevin equations can be used for modeling and understanding of the economical complexity locally or globally.

Kewwords: Economical Dynamics, Time series analysis, Tsallis non-extensive statistics, Fractal Dynamics, Non-equilibrium dynamics, Turbulence.

1.Introduction

Economic dynamics from physical point of view can be regarded as spatial distributed dynamics and connected to the general category of nonlinear distributed systems. Analysis of economical time series show underlying complex and chaotic dynamics in the phase space. Taken's theorem (with the method of delays) permits the reconstruction of an equivalent topologically to the original phase space, which preserves significant geometrical and dynamical properties such as degrees of freedom, fractal dimension, multifractality, Lyapunov exponents, prediction matrix etc. Thus, the reconstructed phase space can be used in order to estimate all the above quantities, as well as phase transitions, statistical behaviors, entropy generation etc. In addition, the phase space can have multifractal properties and intermittent turbulence characteristics that indicate the existence of long range interactions in space and time, as well as the interaction in many scales. These characteristics also indicate the existence of fractional dynamics in the phase space which can be described through fractional Fokker-Planck equations and anomalous diffusion equations. The solutions of these equations are fractional spatiotemporal functions and non-Gaussian distributions functions which fall into the category of Levy distributions and Tsallis distributions. The nonequilibrium stationary states of economical dynamics derive from processes of strong self organization which corresponds to local maxima of Tsallis entropy, while the changes in the control parameters of an economical system can induce a phase transition and a shift of the economical dynamics to new metaequilibrium steady states of maximum Tsallis entropy. This phase transition leads to a multifractal change in the formation of the phase space and an alteration in the phenomenology of the economical system. Finally, the statistics of the dynamics in the multifractal phase space can be described by Tsallis distribution functions of power law and heavy tails form, which can be used for improved prediction methods. In this study, we present complexity theory in relation to econophysics, a summary of an extended nonlinear time series algorithm, as well as new theoretical concepts for the theoretical interpretation and management of economic complexity and crises which take place in local or global level.

Complexity theory considers that the current global economical system can act and evolve as an autonomous system, open to its environment, including multiscale local or global interactions. The physical as well as the anthropogenic environment can be incorporated in the economical system as influences – effects in the form of noise and/or with more intense forms such as external

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control driven processes. The culture state of each period, including psychological, philosophical, political, religious and other ideological states and influences, consists also another one complex anthropogenic system which interacts directly with the economical system. This feedback and non-linear relation of the anthropogenic culture system with the economical system gives rise to a dynamical coupling, reminiscent of the atmosphere and climate dynamics, of the global network of cities and countries etc. Namely, there is an "economical pollution" of the anthropogenic environment from the side effect of the economical system, which is similar to the pollution of the physical environment from the products of the infrastructure (factories etc) of the advanced countries. Similar to the physical environment, the economical system can self-evolve absorbing in a significant degree the outer "pollution" exploiting new functional forms, in order to maintain its autonomy and constitution. This is because the economical system has an advanced complexity with similar characteristics to physical complexity, even though it is not a pure physical system. From this point of view, it is reasonable for someone to "transfer" into the theoretical description and understanding of economical complexity terms from the description of biological complexity and intelligence. This is in accordance to the general scheme of scientific methodology enriched by Complexity theory.

According to the previous paragraph, in this paper we will try to transfer from knowledge from the physical complexity to economical complexity. Of course, this study is very introductive and general and aims to present various terms which are used to the general science of complex systems and can be used for the study of theoretical economics and economical systems. In the following we present the theory of Complexity adapted to economical complexity (paragraph 2), while in paragraph 3 we summarize the algorithm of nonlinear time series analysis. In paragraph 4 we present some results concerning the analysis of two Stock time series and we introduce new theoretical concepts concerning economical complexity in relation to Tsallis statistics, fractal topology, phase transitions, fractional dynamics, percolation, anomalous diffusion etc. Finally, in paragraph 5 we present the conclusions of this study.

2. Theoretical Concepts

In the following we present some aspects of Complexity theory which were developed till today in the physical sciences and as we believe could help for a new theoretical understanding of economical complexity, based on the general and universal theory of self-organization. Already, many papers concerning economical complexity are pointing to this direction [1-6]. Even though economical complexity does not include physical interactions with energy transfer and local interactions through forces, however analogous local interactions through information transferring render economical complexity as a specific type of generalized physical complexity. At this point, we should emphasize the importance of mathematical theory to the physical as well as to any other form of complexity. Nonlinear mathematics including functional analysis, fractals, fractal functions, and fractal topology, as well as nonlinear equations, differential or fractional, can produce stochastic and multiscale processes independent of the region of their interpretation. The existence of the dynamical phase space and the complex evolution in it depends upon the general mathematical characteristics of the system independently of its special character. This character makes the phase space as the tool of unification of every complex process which can be described mathematically .That is the mathematical essence which can produce, or more extremely, can cause the observed complexity in a self consistent way of mathematical forms and any other kind of observed form. These concepts lead us to the deep fundamentals of the mathematical theory including any other mode of scientific theory and observed correlations either at physical or anthropogenic systems.

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2.1 Complexity as a New Physical Theory

Is Complexity a major revolution in science such as Relativity and Quantum Theory? For many scientists attempts to explain complexity and self- organization by using the basic laws of physics have met with little success. Novel forms of self-organization are generally unexpected for the classic reductionistic point of view. However, while complexity is considered as a new and independent physical theory which was developed after the Relativity Theory and Quantum mechanics, it must be consistent with these theories. It is related to far from equilibrium dynamics and concerns the creation and destruction of spatiotemporal patterns, forms and structures. According to Nicolis and Prigogine [7], complexity theory corresponds to the flow and development of space-time correlations instead of the fundamental local interactions. According to the classical point of view the physical phenomena (macroscopic or microscopic) must be reducible to a few fundamental interactions. However, since 1960 an increasing amount of experimental data challenging this idea has become available. This imposes a new attitude concerning the description of Nature. Moreover, according to Sornette [8], systems with a large number of mutually interacting parts and open to their environment can self-organize their internal structure and their dynamics with novel and sometimes surprising macroscopic emergent properties. These general characteristics make the complexity theory, a fundamentally probabilistic theory of the non-equilibrium dynamics.

2.2 Complexity as a Form of Macroscopic Quanticity

The central point of complexity theory is the possibility for a physical system, which includes a great number of parts or elements, to develop internal long-range correlations leading to macroscopic ordering and coherent patterns. These long-range correlations can also appear at the quantum level. In particular, according to the general entanglement character of the Quantum theory, the quantum mechanical states of a system with two or more parts cannot be expressed as the conjunction of quantum states of the separate parts. This situation generally reflects the existence of non-local interactions and quantum correlations while the measurements bearing on either part correspond to random variables which are not independent and can be correlated independently with the spatial distance of the parts [9]. This means that the quantum density operator cannot be factored while the quantum state corresponds to the global and undivided system. The macroscopic manifestation of the quantum possibility for the development of long-range correlations is the spontaneous appearance of ordered behavior in a macroscopic system, examples of which are phenomena like super-fluidity and superconductivity or lasers.

These quantum phenomena display coherent behavior involving the collective cooperation of a huge number of particles or simple elements and a vast number of degrees of freedom. They correspond also to equilibrium or nonequilibrium phase transition processes which constitute the meeting point of quantum theory and complexity. Here the development of quantum long- range correlations leads to a macroscopic phase transition process and macroscopic ordering. It is not out of logic or physical reality to extend the (unifying) possibility of quantum process to developed long-range correlations, according to the quantum entanglement character, into a macroscopic self-organizing factor causing also the far-from equilibrium symmetry breaking and macroscopic pattern formation. From this point of view we can characterize complexity as a form of a macroscopic quanticity [9].

2.3 Complexity Theory and the Cosmic Ordering Principle

The conceptual novelty of complexity theory embraces all the physical reality from equilibrium to non-equilibrium states. This is stated by Castro [10] as follows: "*…it is reasonable to suggest that there must be a deeper organizing principle, from small to large scales, operating in nature which might be based in the theories of complexity, non-linear dynamics and information theory in which dimensions, energy and information are intricately connected*". Tsallis non-extensive statistical theory [11] can be used for a comprehensive description of complex physical systems, as recently we became aware of the drastic change of fundamental physical theory concerning physical systems far from equilibrium.

The dynamics of complex systems is one of the most interesting and persisting modern physical problem, including the hierarchy of complex and self-organized phenomena such as: anomalous diffusion – dissipation and strange kinetics, fractal structures, long range correlations, far from equilibrium phase transitions, reduction of dimensionality, intermittent turbulence etc [12].

For complex systems near equilibrium the underlying dynamics and the statistics are Gaussian as they are caused by a normal Langevin type stochastic process with a white noise Gaussian component. The normal Langevin stochastic equation corresponds to the probabilistic description of dynamics related to the well-known normal Fokker – Planck equation. For Gaussian processes only the moments-cumulants of first and second order are non-zero, while the central limit theorem (CLT) inhibits the development of long range correlations and macroscopic self-organization, since any kind of fluctuation quenches out exponentially to the normal distribution. Also at equilibrium, the dynamical attractive phase space of the distributed system is practically infinite dimensional as the system state evolves in all dimensions according to the famous ergodic theorem of Boltzmann – Gibbs statistics. However, according to Tsallis statistics, even for the case of Gaussian process, the non-extensive character permits the development of long range correlations produced by equilibrium phase transition multi-scale processes.

Generally, the experimental observation of a complex system presupposes non-equilibrium process of the physical system which is subjected to observation, even if the system lives thermodynamically near to equilibrium states. Also, experimental observation includes discovery and ascertainment of correlations in space and time, since the spatiotemporal correlations are related to or caused by the statistical mean values fluctuations. The theoretical interpretation and prediction of observations as spatial and temporal correlations – fluctuations is based on statistical theory which relates the microscopic underlying dynamics to the macroscopic observations, indentified to statistical moments and cumulants. Moreover, it is known that statistical moments and cumulants are related to the underlying dynamics by the derivatives of the partition function to the external source variables [13]. From this point of view, the main problem of complexity theory is how to extend the knowledge from thermodynamical equilibrium states to the far from equilibrium physical states. The non-extensive statistics introduced by Tsallis [11], as the extension of Boltzmann – Gibbs equilibrium statistical theory, is the appropriate base for the non-equilibrium extension of complexity theory. The far from equilibrium statistics can produce the partition function and the corresponding moments and cumulants, in correspondence with Boltzmann – Gibbs statistical interpretation of thermodynamics.

The observed miraculous consistency of physical processes at all levels of physical reality, from the macroscopic to the microscopic level, as well as the inefficiency of existing theories to produce or to predict this harmony and hierarchy of structures inside structures from the macroscopic or the microscopic level of cosmos reveals the nessecity of new theoretical approaches. In this direction, in his book "Randomicity" T. Tsonis [14] presents a significant sinthesis of holistic and reductionistic (analytic) shientific approach. The word randomnicity includes both meanings: chance (randomness) and memory (determinism). According to the fractal generalization of dynamics and statistics we maintain the continuity of functions but abolish their differentiable character based on the fractal calculus which is the non-differentiable generalization of differentiable calculus. At the same time, the deeper physical meaning of fractal calculus is the unification of microscopic and macroscopic dynamical theory based at the space – time fractality [15]. Also, the space-time is related to the fractality – multi-fractality of the dynamical phase – space, which can be manifested as non-equilibrium complexity and self-organization.

After all, we conjecture that the macroscopic self-organization connected with the novel theory of complex dynamics, as it can be observed at far from equilibrium dynamical physical states, is the macroscopic emergence of the microscopic complexity which can be enlarged as the system arrives at bifurcation or far from equilibrium critical points. That is, far from equilibrium the observed physical self-organization manifests at the globally active ordering principle are in priority from local interactions processes. We could conjecture that the concept that local interactions themselves are nothing else than a local manifestation of the universal and holistically active ordering principle, is not far from truth. Namely, what until now is known as fundamental physical laws is nothing else than the equilibrium manifestation or approximation of a universal and globally active ordering principle. This conjecture concerning the fractal unification of macroscopic and microscopic dynamics can be strongly supported by the Tsallis nonextensive q-statistics theory which is verified almost everywhere from the microscopic to the macroscopic level. From this point of view it is reasonable to support that the q-statistics and the fractal generalization of the system's dynamics is the appropriate framework for the description of non-equilibrium complexity. Furthermore, this generalization of Boltzmann – Gibbs statistical mechanics and Newtonian dynamics can be the base of the physical theory for the scientific interpretation of the behavior of many other physical systems as we show in this study.

2.4 Chaotic Dynamics and Statistics

The macroscopic description of complex systems can be approximated by non-linear partial differential equations of the general type:

$$\frac{\partial \vec{U}(\vec{x},t)}{\partial t} = \vec{F}(\vec{u},\vec{\lambda}) \qquad (1)$$

where $\vec{u}$ belongs to a infinite dimensional state (phase) space which is a Hilbert functional space. Among the various control parameters, the Reynold's number is the one which controls the quiet- static or the turbulent states. Generally, the control parameters measure the distance from the thermodynamical equilibrium as well as the critical or bifurcation points of the system for given and fixed values, depending upon the global mathematical structure of the dynamics. As the system passes its bifurcation points a rich variety of spatio-temporal patterns with distinct topological and dynamical profiles can be emerged such as: limit cycles or torus, chaotic or strange attractors, turbulence, vortices, percolation states and other kinds of complex spatiotemporal structures [16].

Generally chaotic solutions of the mathematical system (1) transform the deterministic form of this equation to a stochastic non-linear stochastic system:

$$\frac{\partial \vec{u}}{\partial t} = \vec{\Phi}(\vec{u},\vec{\lambda}) + \vec{\delta}(\vec{x},t) \qquad (2)$$

where $\vec{\delta}(\vec{x},t)$ corresponds to the random fields produced by strong chaoticity [17]. The variables $\vec{u}$ describe various variables which characterize economical dynamics as a field in the economical space, which can present different dynamics and phenomena.

These forms of the non-linear mathematical systems (1,2) correspond to the original version of the new science known today as complexity science. This new science has a universal character, including an unsolved scientific and conceptual controversy which is continuously spreading in all directions of the mathematical descriptions of the physical reality concerning the integrability or computability of the dynamics. The concept of universality was supported by many scientists, after the Poincare's discovery of chaos and its non-integrability, as is it shown in physical sciences in many regions of the physical sciences by the work of Prigogine, Nicolis, and others [16, 18]. Moreover, non-linearity and chaos are the top of a hidden mountain including new physical and mathematical concepts such as fractal calculus, p-adic physical theory, non-commutative geometry, fuzzy anomalous topologies, fractal space-time etc [19]. These novel physical-mathematical concepts obtain their physical power when the physical system lives far from equilibrium.

Furthermore and following the traditional point of view of physical science, we arrive at the central conceptual problem of complexity science. That is, how is it possible the local interactions in a spatially distributed physical system can cause long range correlations or how they can create complex spatiotemporal coherent patterns, as the previous non-linear mathematical systems reveal, when they are solved arithmetically, or as the analysis of in situ observations of physical systems shows.

This question is important for the physical complexity as well as for the economical complexity. The answer of the Complexity theory in this question is based on the principle that there are no fundamental laws neither processes but the dynamics is manifested simultaneously in all scales. This is the basic feature of the complex dynamics which shows the holistic character of the complex systems.

For non-Gaussian processes it is possible for long range correlations to be developed as the cumulants of higher than two order are non-zero [13]. This is the deeper meaning of non-equilibrium self-organization and ordering of complex systems. The characteristic function of the dynamical stochastic field system is related to the partition functions of its statistical description, while the cumulant development and multipoint moments generation can be related with the statistical hierarchy of the statistics. For dynamical systems near equilibrium only the second order cumulants are non-vanishing, while far from equilibrium field fluctuations with higher – order non-vanishing cumulants can be developed.

Finally, using previous descriptions we can now understand how the non-linear dynamics correspond to self-organized states as the high-order (infinite) non-vanishing cumulants can produce the non-integrability of the dynamics. From this point of view, the linear stochastic dynamics is inefficient to produce the non-Gaussian, holistic (non-local) and self-organized complex character of non-equilibrium dynamics. That is, far from equilibrium complex states can be developed including long range correlations with non-Gaussian distributions of their dynamic variables.

2.5 Strange attractors and Self-Organization

In this section we present the general theory of multifractal structures which describe the economical dynamics in the state space as a multiscale process. In particular, when the dynamics is strongly nonlinear then far from equilibrium it is possible to occur strong self-organization and intensive reduction of dimensionality of the state space, by an attracting low dimensional set with parallel development of long range correlations in space and time. The attractor can be periodic (limit cycle, limit m-torus), simply chaotic (mono-fractal) or strongly chaotic with multiscale and multifractal profile as well as attractors with weak chaotic profile known as SOC states. This spectrum of distinct dynamical profiles can be obtained as distinct critical points (critical states) of the nonlinear dynamics, after successive bifurcations as the control parameters change. The fixed points can be estimated by using a far from equilibrium renormalization process as it was indicated by Chang [20].

From this point of view phase transition processes can be developed between different critical states, when the order parameters of the system are changing. The far from equilibrium development of chaotic (weak or strong) critical states include long range correlations and multiscale internal self organization. Now, these far from equilibrium self organized states, cause the equilibrium BG statistics and BG entropy, to be transformed and replaced by the Tsallis extension of $q-$ statistics and Tsallis entropy. The extension of renormalization group theory and critical dynamics, under the $q-$ extension of partition function, free energy and path integral approach has been also indicated [11]. The multifractal structure of the chaotic attractors can be described by the generalized Rényi fractal dimensions:

$$D_{\bar{q}} = \frac{1}{\bar{q}-1} \lim_{\lambda \to 0} \frac{\log \sum_{i=1}^{N\lambda} p_i^{\bar{q}}}{\log \lambda} \tag{3}$$

where $p_i \sim \lambda^{\alpha(i)}$ is the local probability at the location (i) of the phase space, λ is the local size of phase space and $a(i)$ is the local fractal dimension of the dynamics. The Rényi $\bar{q}$ numbers (different from the $q-$ index of Tsallis statistics) take values in the entire region ($-\infty, +\infty$) of real numbers. The spectrum of distinct local pointwise dimensions $\alpha(i)$ is given by the estimation of the function $f(\alpha)$ defined by the scaling of the density $n(a,\lambda) \sim \lambda^{-f(a)}$, where $n(a,\lambda)da$ is the number of local regions that have a scaling index between a and $a+da$. This reveal $f(a)$ as the fractal dimension of points with scaling index a. The fractal dimension $f(a)$ which varies with a shows the multifractal character of the phase space dynamics which includes interwoven sets of singularity of strength a, by their own fractal measure $f(a)$ of dimension [21]. The multifractal spectrum $D_{\bar{q}}$ of the Renyi dimensions can be related to the spectrum $f(a)$ of local singularities in the phase space of the complex dynamics.

It is also known the Renyi's generalization of entropy according to the relation: $S_q = \frac{1}{q-1} \log \sum_i P_i^q$. However, the above description presents only a weak or limited analogy between multifractal and thermodynamical objects. The real thermodynamical character of the multifractal objects and multiscale dynamics was discovered after the definition by Tsallis [11] of the $q-$ entropy related with the $q-$ statistics as it is summarized in the next section**.** As Tsallis has shown Renyi's entropy as well as other generalizations of entropy cannot be used as the base of the non-extensive generalization of thermodynamics.

2.6 Universality of Tsallis non-extensive statistical mechanics

According to Tsallis, Boltzmann-Gibbs statistical mechanics and standard thermodynamics do not seem to be universal. Tsallis extended the Boltzmann- Gibbs statistics and Boltzmann-Gibbs entropy to non-extensive statistical me-chanics and non-extensive *q*-entropies. The classical Boltzmann-Gibbs extensive thermostatistics constitutes a powerful tool when microscopic interactions and memory are short ranged and space-time is a continuous and differentiable Euclidean manifold. However, far from equilibrium these characteristics are changed as multiscale coupling and non-locality characteristics appear. In turbulence for example, the presence of long-range correlations imply non-local interactions between large and small scales as the relation between them is not local in space and time but functional. This indicates that small-scale fluctuations in each time space point depend on the large scale motions in the whole time-space domain and vice versa. Generally, the non-extensive statistical mechanics introduced by Tsallis rather than being just a theoretical construction it is relevant to many complex systems at the macroscopic or the microscopic level with long-range correlations-interactions or multifractal behavior. A crucial property of Tsallis entropy S_q is the pseudo-additivity for given subsystems A and B in the sense of factorizability of the microstate.

2.6.1 The Highlights of Tsallis Theory.

In our understanding, Tsallis theory, which is more than a simple generalization of thermodynamics for chaotic and complex systems or a non-equilibrium generalization of BG statistics, can be considered as a strong theoretical foundation for the unification of macroscopic and microscopic physical complexity. From this point of view, Tsallis statistical theory is the other side of the fractional generalization of dynamics, while its essence is merely the efficiency of nature for its self-organization and the development of long-range correlations of coherent structures in complex systems.

2.6.2 Non-extensive entropy (S_q)

Any extension of physical theory is usually related to some special type of mathematics. Non-extensive Tsallis statistical theory is connected to the *q*-extension of exponential and logarithmic functions and the *q*-extension of a Fourier transform (FT) [11].

Tsallis, inspired by multi-fractal analysis proposed that the BG entropy

$$S_{BG} = -k \sum p_i \ln p_i = k < \ln(1/p_i) > \tag{4}$$

cannot describe all of the complexity of non-linear dynamic systems. BG statistical theory presupposes ergodicity of the underlying dynamics in the system phase space. The complexity of dynamics is far beyond simple ergodic complexity and can be described by non-extensive Tsallis statistics based on the extended concept of *q*-entropy:

$$S_q = k\left(1 - \sum_{i=1}^{N} p_i^q\right) / (q-1) = k < \ln_q(1/p_i) > \tag{5}$$

For a continuous state space, we have

$$S_q = k\left[1 - \int [p(x)]^q \, dx\right] / (q-1) \tag{6}$$

For a system with short-range correlations the Tsallis *q*-entropy S_q asymptotically leads to BG entropy (S_{BG}) corresponding to *q* = 1. For probabilistically dependent or correlated system *A* and *B*, it can be proven that

$$S_q(A+B) = S_q(A) + S_q(B/A) + (1-q)S_q(A), \tag{7}$$

where
$S_q(A) \equiv S_q\left(\left\{p_i^A\right\}\right)$, $S_q(B) \equiv S_q\left(\left\{p_i^B\right\}\right)$, $S_q(B/A)$ and $S_q(A/B)$ are the conditional entropies of systems A, B. When the systems are probabilistically independent, then relation (7) changes to

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \quad (8)$$

The first part of $S_q(A+B)$ is additive ($S_q(A)+S_q(B)$) while the second part is multiplicative including long – range correlations supporting the macroscopic ordering phenomena. According to Tsallis, if the correlations are either strictly or asymptotically nonexistent, the BG entropy is extensive, whereas S_q for $q \neq 1$ is non extensive. Oppositely, for strong correlated states BG entropy is non-extensive while for a special q-value S_q is extensive.

2.6.3 The *q*- extension of statistics

According to the Tsallis q-extension of the entropy principle, any stationary random variable can be described as the stationary solution of a generalized fractional diffusion equation. For metastable stationary solutions of a stochastic process, the maximum entropy principle of BG statistical theory can faithfully be described by the maximum (extreme) of the Tsallis q-entropy function. Extremization of Tsallis q-entropy corresponds to the q-generalized form of the normal distribution function

$$p_q(x) = A_q\sqrt{\beta}e_q^{-\beta(x<x>_q)^2} \quad (9)$$

where $A_q = \sqrt{(q-1)/\pi}\Gamma(1/(q-1))/\Gamma((3-q)/[2/(q-1)])$ for $q > 1$,
and $A_q = \sqrt{(1-q)/\pi}\Gamma((5-3q)/[2(1-q)])/\Gamma((2-q)/(1-q))$ for $q < 1$, $\Gamma(z)$ being the Riemann function.

The q-extension of statistics also includes q-extension of the central limit theorem, which can faithfully describe non-equilibrium long-range correlations in a complex system. The normal central limit theorem concerns Gaussian random variables (x_i) for which the sum $Z = \sum_{i=1}^{N} x_i$ gradually tends to a Gaussian process as $N \to \infty$, while its fluctuations tend to zero, in contrast to the possibility of non-equilibrium fluctuations with long-range correlations. Using the FT q-extension, we can prove that q-independence means independence for q = 1 (normal central limit theorem), but for $q \neq 1$ it means strong correlation (q-extended central limit theorem). In this case ($q \neq 1$), the number of allowed states $W_{A_1+A_2+\ldots+A_N}$ in a system composed of $(A_1, A_2, \ldots, A_N)$ subsystems is expected to be less than $W_{A_1+A_2+\ldots+A_N} = \Pi_{i=1}^N W_{A_i}$ where $W_{A_1}, W_{A_2}, \ldots, W_{A_N}$ are the possible states of the subsystems. This means self organization of dynamics for $q \neq 1$ and development of long range correlations in space and time.

3. Extended Algorithm of Nonlinear analysis of economical data

In general, far from equilibrium spatially extended dissipative physical systems reveal self organized complexity and chaotic dynamics or other complex dynamics at the edge of chaos, as well as non–equilibrium statistical profile according to Tsallis non–extensive statistical theory [11]. The attempt to understand the complex deterministic motion of spatially extended nonlinear dissipative systems, so–called spatiotemporal chaos (STC), is at the forefront of research in nonlinear dynamics. In contrast to simple chaotic systems in the time domain, in which few degrees of freedom are nonlinearly coupled, spatially extended systems include an infinite number of spatially distributed degrees of freedom. Spatiotemporal chaos involves intermediate situations between chaos and turbulence or to fully developed turbulence when the system is sufficiently confined. In these states it is possible to characterize the dynamics from a local time series alone estimating fractal dimensions, or Lyapunov exponents in the reconstructed phase space. Moreover, defect turbulence and intermittent turbulence, self organized criticality (SOC), avalanche threshold dynamics, spinodal and nucleation phenomena and far from equilibrium phase transition, Tsallis entropies and non–Gaussian fluctuations as well as diffusion or Levy motion, are some of the different manifestations of spatiotemporal complexity and multiscale – multifractal phenomena that must be studied using nonlinear signal analysis [22-24].

The universal economical system includes local sub-systems and therefore can be regarded as a complex spatiotemporal distributed system. From this point of view, we can study the spatiotemporal dynamics of an economical system with the same way that we study the physical complex distributed systems. In this paragraph we present a possible algorithm of nonlinear time series analysis of economical data.

3.1. Analysis at the domain of time

(a) Autocorrelation Coefficient and Power Spectrum (Linear correlations, periodicities, scaling laws)
(b) (Mutual Information (Linear and Nonlinear Correlations)
(c) Probability Distributions (Power Laws)
(d) Hurst exponent (Persistence, anti-persistence, white noise)
(e) Flatness Coefficient F (Intermittent turbulence)
(f) Structure Functions (Turbulence, anomalous diffusion)
(g) Phase portrait (Low Dimensionality)
(h) Entropy, energy, multifractal structures
(i) Estimation of q-Tsallis Statistics
(j) Wavelet analysis (Spatiotemporal structures)

3.2 Computation of Geometrical characteristics in the reconstructed state space

(a) Correlation Dimension (Degrees of freedom)
(b) Generalized Dimension (Multifractals)

(c) False Neighbors (Degrees of Freedom)
(d) Singular values spectrum (SVD components, filtering)

3.3 Computation of Dynamical Characteristics in the reconstructed state space
(a) Maximum Lyapunov Exponent (Sensitivity in initial conditions)
(b) Power Spectrum of Lyapunov Exponents (Sensitivity in initial condi-
tions in all dimensions in space state)
(c) Nonlinear modeling and nonlinear prediction algorithms

3.4 Testing of Null Hypothesis in order to discriminate between low dimensional chaotic dynamics and linear high dimensional stochastic dynamics
(a) Surrogate data
(b) Discriminating statistics

3.5 Singular Value Analysis
(a) Estimate Degrees of Freedom
(b) Filter signals from White or Colored Noise
(c) Search for input-output dynamics

3.6 Recently in the above algorithm significant new tools were added
(a) Fuzzy analysis of time series
(b) Cellular automata, genetic algorithms and neural network modeling
(For spatiotemporal modeling and prediction of complexity)

4. Tsallis *q*-triplet of economical data and theoretical framework

Iliopoulos et al. [25] estimated statistical features of two time stock market time series, Standart & Poor's 500 (S & P) 500 and TVIX, based on the previously described algorithm and on Tsallis statistics. In summary, the results showed that the statistics of the dynamics in the multifractal phase space can be described by Tsallis distribution functions of power law and heavy tails forms, related to the existence of long range interactions in space and time, as well as the interaction in many scales. In addition, the existence of non-equilibrium phase transitions were found depicted clearly in the variations of Tsallis *q*-triplet values, which correspond to multi-fractal changes in the formation of the phase state and an alteration in the phenomenology of the economical system. Finally, the results revealed that underline the inefficiency of classical statistical theories based on the classical central limit theorem to explain the complexity of the economic system dynamics, since these theories include smooth and differentiable spatial-temporal functions or Gaussian statistics (Boltzmann-Maxwell statistical mechanics). On the contrary, the results of this study indicate the presence of non-Gaussian non-extensive statistics with heavy tails probability distribution functions, which are related to the *q*-extension of central limit theorem. The afore mentioned results can be better understood in the framework of modern theoretical concepts concerning non-extensive statistical mechanics [11], fractal topology [26], turbulence theory [27], strange dynamics [15], percolation theory [28], anomalous diffusion theory and anomalous transport theory [29-30], fractional dynamics [31] and non-equilibrium RG theory [20].

4.1 The *q*-extension of Central Limit Theorem (CLT) and the *q*-triplet of Tsallis

The results of this study showed clearly the non-extensive character of the economical system and the multi-scale strong correlations from the microscopic to the macroscopic level indicating the inefficiency of classical statistical theories, based on the classical CLT, to explain the complexity of the economical system dynamics. According to the classical CLT the probability density function of a sum of independent random variables is Gaussian, even when the single variables have long range tails. However, the non-Gaussian with heavy tails probability distribution functions, which were observed in the economical system statistics, are related to the *q*-extension of CLT.

Tsallis non-extensive statistical mechanics includes the *q*-generalization of the classic CLT as a *q*-generalization of the Levy – Gnedenko central limit theorem. Umarov et al. [32] applied for globally correlated random variables. The *q*-generalization of CLT based at the *q*-Fourier transform of a *q*-Gaussian can produce an infinite sequence (q_n) of *q*-parameters by using the function $Z(s)=\frac{1+s}{3-s}$, $s\in(-\infty,3)$ and its inverse $z^{-1}(t), t\in(-1,\infty)$. It can be shown that $z(1/z(s))=1/s$ and $z(1/s)=1/z^{-1}(s)$, as well as if $q_1=z(q)$ and $q_{-1}=z^{-1}(q)$ it follows that:

$$z\left(\frac{1}{q_1}\right)=\frac{1}{q}, z\left(\frac{1}{q}\right)=\frac{1}{q-1} \text{ and } q_{-1}+\frac{1}{q_1}=2.$$

The *q*-generalization of the CLT consistent with non-extensive statistical mechanics is as follows:
For a sequence $q_k, k\in¢$ with $q_k\in[1,2]$ and a sequence $x_1, x_2, x_N, \ldots$ of q_k -independent and identically distributed random variable then the $z_n=x_1+x_2+x_N+\ldots$ is also a q_{k-1}-normal distribution as $N\to\infty$, with corresponding statistical attractor $G_{q_{k-1}}(\beta_{k_j}x)$.

The *q*-independence corresponds to the relations:

$$F_q(x+y)(\xi)=F_q[x](\xi)\otimes_q F_q[y](\xi) \tag{10}$$

$$F_{q-1}(x+y)(\xi)=F_{q-1}[x](\xi)\otimes_q F_q[y](\xi) \tag{11}$$

where $q=z(q_{-1})$. The *q*-independence means independence for $q=1$ and strong correlation for $q\neq 1$ [11].
The *q*-CLT states that an appropriately scaled limit of sums of q_k correlated random variables is a q_{k-1} – Gaussian, which is the q_k^* -Fourier image of a q_k^* - Gaussian. The q_k, q_k^* are sequences:

$$q_k=\frac{2q+k(1-q)}{2+k(1-q)} \text{ and}$$

$$q_k^* = q_{k-1} \tag{12}$$

for $k = 0, \pm 1, \pm 2, ...$

including the triplet (P_{att}, P_{cor}, P_{scl}), where P_{att}, P_{cor} and P_{scl} are parameters of attractor, correlation and scaling rate respectively and corresponds to the q-triplet (q_{sens}, q_{rel}, q_{stat}) according to the relations[32] :

$$(P_{att}, P_{cor}, P_{scl}) \equiv (q_{k-1}, q_k, q_{k+1}) \equiv (q_{sens}, q_{rel}, q_{stat}) \tag{13}$$

The parameter $P_{att} \equiv q_{sens} \equiv q_{k-1}$ describes the non-ergodic q-entropy production of the multiscale correlated process as the system shifts to the state of the q_{att} – Gaussian, where the q-entropy is extremized in accordance with the generalization of the Pesin's theorem [11]:

$$K_{qsen} \equiv \lim_{t\to\infty} \lim_{W\to\infty} \lim_{M\to\infty} \frac{S_q(P_i(t))/k}{t} = \lambda_{qsen} \tag{14}$$

The parameter $P_{cor} \equiv q_{rel} \equiv q_k$ describes the q-correlated random variables participating to the dynamical process of the q-entropy production and the relaxation process toward the stationary state.

The parameter $P_{slc} \equiv q_{stat} \equiv q_{k+1}$ describes the scale invariance profile of the stationary state corresponding to the scale invariant q-Gaussian attractor as well as to an anomalous diffusion process mirrored at the variance scaling according to general, asymptotically scaling, from:

$$N^D P_x(x) \sim G\left(\frac{x}{N^D}\right) \tag{15}$$

Where $P_x(x)$ is the probability function of the self similar statistical attractor G and D is the scaling exponent characterizing the anomalous diffusion process [33]:

$$\left\langle x^2 \right\rangle \sim t^{2D} \tag{16}$$

The non-Gaussian multi-scale correlation can create the intermittent multi-fractal structure of the phase space mirrored also in the physical space multi-fractal distribution of the turbulent dissipation field. The multi-scale interaction at critical non-equilibrium steady states (NESS) creates the heavy tail and power law probability distribution function obeying the q-entropy principle. The singularity spectrum of a critical NESS corresponds to extremized Tsallis q-entropy.
In this framework of theoretical modeling of distributed random fields, the fractal – multifractal character of the economical system at every NESS as it was shown in this study, indicate the existence of physical local singularities for the spatial-temporal distribution of economical system variables. The singularity behavior of a given random field function $f(x)$ at a given point x_* is defined as the greatest exponent h so that f is Lipschitz h at x_* [34]. The Hölder exponent $h(x_*)$ measures how irregular f is at the point X_* according to the relation:

$$\left| f(x) - P(x - x_*) \right| \le C \left| x - x_* \right|^h \tag{17}$$

The smaller the exponent $h(x_*)$ is the more irregular (singular) is the function f at the point x_*. On a self similar fractal set the singularity strength of the measure μ at the point x is described by the scaling relation:

$$\mu(B_x(\varepsilon)) ; \int_{B_x(\varepsilon)} d\mu(y) \sim \varepsilon^{a(x)} \tag{18}$$

Where $B_x(\varepsilon)$ is a ball of size (ε) centered at x and $\mu(B_x)$ is the fractal mass in the B_x region. Homogeneous measures are characterized by a singularity spectrum supported by a single point (a_0, $f(a_0)$) while multifractal sets involve singularities of different strengths (α) described by the singularity spectrum $f(\alpha)$.
The singularity spectrum is defined as the Hausdorff dimension of the set of all points x such that $a(x) = a$. The singularity strength $a(x)$ of the fractal mass measure is related to the local fractal dimension of the fractal set. More generally, the singularity spectrum associated with singularities h (Hölder exponents) of a distributed random variable is given follows:

$$D(h) = d_H \left\{ x : h(x) = h \right\} \tag{19}$$

That is $D(h)$ is the Hausdorff dimension of the set of all points x such as $h(x)=h$. For the economical system the singularity spectrum $D(h)$ corresponds to the scaling invariance of the description according to the scaling transformation of the equations:

$$r \to r'\lambda^{-1}, u_r \to u_{r'}\lambda^{-h}, b_r \to b_{r'}\lambda^{-h} \tag{20}$$

where h is a free parameter and u_r, b_r are variables (e.g u_r is "velocity" and b_r is "field") of the economical system. The $h(x)$ singularity exponent of the fractal mass measure and the $F(\alpha)$ singularity spectrum describe the scaling of the "energy" dissipation in the turbulent economical system.

4.2 Fractional Calculus

The fractal-multifractal structure of the economical system indicates the generalization of the dynamical differential equations for the description of the economical dynamics to the fractional dynamics of the economical system, since the functions of the distributed physical system's variables are irregular and they are produced by fractional dynamics on fractal structures. The differentiable nature of magnitudes with smooth distributions of the macroscopic picture of the economical processes is a natural consequence of the Gaussian microscopic randomness which, through the classical CLT, is transformed to the macroscopic, smooth and differentiable processes. The classical CLT is related to the condition of microscopic and macroscopic time-scale separation, where at the long-time limit the memory of the microscopic non-differentiable character is lost. On the other hand, the q-extension of CLT induces the nonexistence of time-scale separation between microscopic and macroscopic scales as the result of multiscale global correlations which produce fractional dynamics and singular functions of spatio-temporal dynamical economical variables.

Fractal sets are measurable metric sets with non-integer Hausdorff dimension. The elements of a fractal set can be represented by n-tuples of real numbers $x=(x_1,x_2,...,x_n)$ such that a fractal set F is embedded in R^n. A fractal function is defined on a fractal set as follows:

$$f(x)=\sum_{i=1}^{\infty}\beta_i X_{E_i}(x) \quad (21)$$

where X_E is the characteristic function of E. The continuous function $f(x)$ is defined as follows:

$\lim_{x\to y} f(x)=f(y)$ whenever $d(x,y)=0$ for the metric $d(x,y)$ defined in R^n and for points $(x,y)\in f$. The Hausdorff measure μ_H of a subset $E\in F$ is defined by:

$$\mu_H(E,D)=W(D)\lim_{d(E_i)\to 0}\inf_{\{E_i\}}\sum_{i=1}^{\infty}\left|d(E_i)\right|^D \quad (22)$$

Where $E\subset U_{i=1}^{\infty}E_i$, D is the Hausdorff dimension of $E\subset U$, $d(E_i)$ are the diameters of $\{E_i\}$ and $W(D)$ for balls $\{E_i\}$ covering F is given by

$$W(D)=\frac{\Pi^{D/2}2^{-D}}{\Gamma\left(D/2+1\right)} \quad (23)$$

The Lebesque – Stieltjes integral over a D-dimensional fractal set of a function $f(x)$ is defined by

$$\int_F f d\mu \equiv \sum_{i=1}^{\infty}\beta_i\mu_H(E_i) \quad (24)$$

And it can be proved to be given by the relation:

$$\int_F f d\mu_H=\frac{2\Pi^{D/2}\Gamma(D)}{\Gamma\left(D/2\right)}\left(I_o^D f\right) \quad (25)$$

Where

$\left({}_zI_x^a f\right)(z)\equiv\frac{1}{\Gamma(a)}\int_z^{\infty}(x-x')^{a-1}f(x')dx'$ is the Riemann Liouville fractional integral [31]. The last relation connects the integral on fractals with fractional integrals and permits the application of different tools of the fractional calculus for the fractal medium. Respectively to the Riemann Liouville fractional integral on a fractal set F we can define the Riemann Liouville fractional derivative by

$${}_zD_x^a f(x)=\frac{1}{\Gamma(n-a)}\frac{\partial^n}{\partial x^n}\int_z^x\frac{f(x')}{(x-x')^{a-n+1}}dx'$$
$$n-1<a<n \quad (26)$$

The nonlocal character is evident in both cases of fractional derivative and integral on a fractal set. The nonlocal character of fractional calculus is related to multiscale and self-similar character of the fractal structure. The fractional extension of integral and differential calculus can be used for the description of the non-local multiscale phenomena described by fractional Equations, or the fractional fractal states, or the fractional Fokker-Planck Equation (fFPE) of fractal media [31]. The solution of the fractional equations, correspond to fractional non-differentiable singular self-similar functions as we can observe at the experimental data. Generally, fractional differential integral equations have as solutions non-differentiable (singular) spatio-temporal distribution functions of physical magnitudes.

4.3 Anomalous diffusion and strange dynamics

In this section we present the significant phenomenon of anomalous diffusion. Normal diffusion is connected to Gaussian dynamics, while the results obtained by the nonlinear time series analysis of economical time series showed non-Gaussian statistics. Thus, it is reasonable to consider that the stochastic dynamics of economics can be related to anomalous diffusion, which generates long range correlations described by the maximization of Tsallis entropy and to Boltzmann – Gibbs – Shannon entropy. In addition, as in physical turbulence phenomena, in economics we can also have phenomena of symmetry scaling which produce the self-organized character of the economical complexity.

In particular, nonlinear dynamics can create fractal structuring of the phase space and global correlations in the nonlinear system. For non-extensive systems the entire phase space is dynamically not entirely occupied (the system is not ergodic), but only a scale-free –like part of it is visited yielding a long-standing (multi)-fractal-like occupation. According to [35] Tsallis entropy can be rigorously obtained as the solution of a nonlinear functional equation referred to the spatial entropies of the subsystems involved including two principal parts. The first part is linear (additive) and leads to the extensive Boltzman-Gibbs entropy. The second part is multiplicative corresponding to the non-extensive Tsallis entropy referred to the long range correlations. The fractal –multifractal structuring of the phase space makes the effective number W_{eff} of possible states, namely those whose probability is nonzero, to be smaller ($W_{\text{eff}} < W$) than the total number of states. This is the statistical manifestation of self-organization process.

According to [15] the topological structure of phase space of nonlinear dynamics can be highly complicated including trapping and flights of the dynamics through a self-similar structure of islands. The island boundary is sticky making the dynamics to be locally trapped and "stickiness". The set of islands is enclosed within the infinite fractal set of cantori causing the complementary features of trapping and flight being the essence of strange kinetics and anomalous diffusion.

The dynamics in the topologically anomalous phase space corresponds to a random walk process which is scale invariant in spatial and temporal self-similarity transform:

$$\hat{R}: t' \to \lambda_t t, \xi' = \lambda_\xi \xi \tag{27}$$

The spatial-temporal scale invariance causes strong spatial and temporal correlations mirrored in singular self-similar temporal and spatial distribution functions which satisfy the fractional generalization of classical Fokker-Planck-Kolmogorov equation (FFPK – equation) [15]:

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha}{\partial(-\xi)^\alpha}(AP) + \frac{1}{2}\frac{\partial^{2\alpha}}{\partial(-\xi)^{2\alpha}}(BP) \tag{28}$$

Where $P \equiv P(\xi, t)$ is the probability density of the state (ξ) at the time (t). The critical components (α, β) correspond to the fractal dimensions of the spatial-temporal non-gaussian distributions of the spatial-temporal functions-processes or probability distributions. The quantities A, B are given by

$$A = \lim_{\Delta t \to 0} \frac{\left\langle\left\langle\left|\Delta\xi^\alpha\right|\right\rangle\right\rangle}{(\Delta t)^\beta}, \mathrm{B} = \lim_{\Delta t \to 0} \frac{\left\langle\left\langle\left|\Delta\xi^{2\alpha}\right|\right\rangle\right\rangle}{(\Delta t)^\beta} \tag{29}$$

where <<…>> denotes a generalized convolution operator [15].

The FFPK equation is an archetype fractional equation of fractional stochastic dynamics in a (multi)-fractal phase space with fractal temporal evolution caused by the self similar and multiscale structure of islands around islands, responsible for the flights and trappings of the dynamics. The "spatial" random variable can be any physical variable at a certain position in physical space etc, underlying to the nonlinear chaotic dynamics. The fractional dynamics includes fractal distribution of variables, as well as fractal distribution of "energy" dissipation field.

The fractional temporal derivative $\partial^\beta / \partial t^\beta$ in equations allows one to take fractal-time random walks into account, as the temporal component of the strange dynamics in fractal-turbulent media. The waiting times follow the power law distribution $P(\tau) \sim \tau^{-(1+\beta)}$ since the "Levy flights" of the dynamics also follow the power law of distribution.

The asymptotics (root mean square of the displacement) of the transport process is given by $\left\langle|\xi|^2\right\rangle = 2D\, t^\mu$, while the generalized transport coefficient μ depends on the values of the fractal coefficients (α, β), according to the relation $\mu = \frac{\beta}{\alpha}$ [29].

The solution of the fractal kinetic equation corresponds to Levy distributions and asymptotically to Tsallis q-Gaussians. The set of points visited by the random walker can reveal a self-similar fractal structure produced by the extremization of Tsallis q-entropy. The q-Gaussian distribution of the fractal structure created by the strange dynamics and the extremized q-entropy asymptotically corresponds to the Levy distribution $P(\xi) \sim \xi^{-1-\gamma}$ where the q-exponent is related to the Levy exponent γ by $q = \frac{3+\gamma}{1+\gamma}$. The Levy exponent γ corresponds to the fractal structure of the points visited by the random walker. According to [11] the fractal extension of dynamics includes simultaneously the q-extension of statistics as well as the fractal extension RNG theory in the fractional Fokker-Planck-Kolmogorov Equation (FFPK).

As far as the economical system dynamics is concerned, the (ξ) variable in equation (28) could correspond to any variable of the economical system, while $P(\xi, t)$ describes the probability distribution of the economical variable. The variables $A(\xi)$, $B(\xi)$ corresponds to the first and second moments of probability transfer and describe the wandering process in the fractal space (phase space) and time. The fractional space and time derivatives $\partial^\beta / \partial t^\beta, \partial^\alpha / \partial x^\alpha$ are caused by the multifractal (strange) topology of phase space which can be described by the anomalous phase space renormalization transform [36]. We must notice here that the multifractal character of the turbulent field in the physical space is the mirroring of the phase space strange topology in the spatial multifractal distribution of the dynamical variables.

The q-statistics of Tsallis corresponds to the meta-equilibrium solutions of the FFPK equation [37]. Also, the metaequilibrium states of FFPK equation correspond to the fixed points of Chang non-equilibrium RNG theory for state space [15, 20]. The anomalous topology of phase space dynamics includes inherently the statistics as a consequence of its multiscale and multifractal character.

From a wider point of view the FFPK equation is a partial manifestation of a general fractal extension of dynamics. According to [37], the Zaslavsky's equation can be derived from a fractional generalization of the Liouville and BBGKI equations. The fractal extension of dynamics is based on the fact that the fractal structure of the spatially distributed system can be replaced by a fractional continuous model. In this generalization the fractional integrals can be considered as approximations of integrals on fractals. Also, the fractional derivatives are related to the development of long range correlations and localized fractal structures.

4.4 Fractal topology, critical percolation and stochastic dynamics

In the following we present the theory of [38] concerning fractal topology and topological phase transitions, which can describe the economical complexity as is manifested in the phase space. This theory is a general description of random fields in the physical as well as in the phase space. In particular, we present some basic concepts concerning topological aspects of percolating random fields, which can be used in order to shed light to the complex and non-extensive character of the economical system.

For any random field distribution $\psi(\vec{x})$ in the n-dimensional space (E^n) there exists a critical percolation threshold which divides the space E^n into two topological distinct parts: Regions where $\psi(\vec{x}) < h_c$ marked as "empty", and regions where $\psi(\vec{x}) > h_c$, marked as "filled". When $\psi(\vec{x}) \neq h_c$, one of these parts will include an infinite connected set which is said to percolate. As the threshold h changes we can find the critical threshold h_c where the topological phase transition occurs, namely the

non-percolating part starts to percolate. The random field may be a spatial distribution of physical random magnitudes or it can correspond to the random distribution of physical properties in the phase space of the underlying dynamics.
The geometry of the percolating set at the critical state ($h \to h_c$) is a typical fractal set for length scales between microscopic distances and percolation correlation length which diverges. The statistically self-similar geometry includes power-law behavior of the "mass" density of the fractal set such as "fractal mass density" ~ x^{D-n}, where x is the length scale, D is the Hausdorff fractal dimension which must be smaller than the dimensionality (n) of the embedding Euclidean space. In addition to the parameter D of the fractal dimension, there is the index of connectivity θ which describes the "shape" of the fractal set and may be different for fractals even with equal values of the fractal dimension D. The index of connectivity θ is defined as characterizing the shortest (geodesic) line connecting two different points on the fractal set by the relation $d_\theta = (2+\theta)/2$, where d_θ is the minimal Hausdorff dimension of the minimal (geodesic) line for all possible homeomorphisms that transform the fractal F into a fractal F′. The geodesic line on a self-similar fractal set (F) is a self affine fractal curve whose own Hausdorff fractal dimension is equal to $(2+\theta)/2$. The index of connectivity plays an essential role in many dynamical phenomena on fractals, while it is a topological invariant of the fractal set F.
From the fractal dimension D and the connectivity index θ we can define a hybrid parameter $d_s = \dfrac{2D}{2+\theta}$ which is known as the spectral or the fracton dimension which represents the density of states for vibrational excitations in fractal network termed as fractons. The root mean square displacement of the random walker on the fractal set is given by

$$\left\langle |\xi|^2 \right\rangle \sim t^{2/2+d_\theta} = t^{1/d_\theta} \tag{30}$$

Where d_θ is the fractal dimension of the self-affine trajectory on the fractal set. Also, the spectral dimension which measures the probability of the random walker to return to the origin, is given by

$$P(t) \sim t^{-\frac{d_s}{2}} \tag{31}$$

while the Hausdorff fractal dimension D is a structural characteristic of the fractal structure F, the spectral dimension d_s mirrors the dynamical properties such as wave excitation, diffusion etc. The fractal dimension d_f of the fractal structure F of a percolating random field distributed in the E^n Euclidian space is given by $d_f = n - \beta/\nu$, where β, ν are the universal critical exponents of the critical percolation state. According to the Alexander-Orbach (AO) conjecture [28], the spectral dimension d_s has been established to be equal to the value d_s=4/3, for all embedding dimensions $n \geq 2$. Especially, for embedding dimensions $2 \leq n \leq 5$, [28] has improved the AO conjecture to the value $d_s = C ;\ 1,327$ where C is the percolation constant. This constant determines the minimal fractional number of the degrees of freedom that the random walker must have to reach the infinitely remote point in the Euclidian embedding space E^n.

At the percolation threshold the probability P that the lattice sites are occupied approach the value P_c $(P \to P_c)$ while the percolation correlation length ξ diverges as $\xi \sim |P-P_c|^{-\nu}$. For $P > P_c$ the probability to belong to the infinite cluster is $P_\infty(P) \sim (P-P_c)^\beta \sim \xi^{-\beta/\nu}$ whereas the dc conductivity behaves as $\sigma_{dc} \sim (P-P_c)^\mu \sim \xi^{-\mu/\nu}$. The critical exponents (β, ν, μ) are universal independent of the type of the percolation problem, depending only on the ambient dimensionality n (lattice embedding dimension in the Euclidean space R^n). The "mass" (M) of a connected cluster of the percolation set scales as $M \sim \xi^n P_\infty(P) \sim \xi^{n-\beta/\nu}$ leading to the nontrivial Hausdorff dimension $d_f = n - \beta/\nu$. The connectivity index θ in the case of a fractal percolating set is given by the relation $\theta = (\mu-\beta)/\nu$, while the anomalous diffusion of a random walker on a connected percolation cluster is described by average traveled distance

$$< r^2(t) > \sim t^{2/2+\theta} \tag{32}$$

The averaging over all connected percolation clusters replaces the previous relation by

$$< r^2(t) > \sim t^{(2-\beta/\nu)/2+\theta} \tag{33}$$

The fractional equations, equations, include fractional time derivatives

$$\frac{\partial^\beta}{\partial t^\beta}(\ldots) = \frac{1}{\Gamma(m-\beta)} \frac{\partial^m}{\partial t^m} \int_0^t \frac{dt'}{(t-t')^{(1+\beta-m)}}(\ldots) \tag{34}$$

and space derivatives:

$$\frac{\partial^a}{\partial x_i^a}(\ldots) = \frac{1}{\Gamma(1-a)} \frac{\partial}{\partial x_i} \int_{-\infty}^{x_i} \frac{dx_i'}{(x_i - x_i')^a}(\ldots) \tag{35}$$

[26].

The parameter (β) has the meaning of the fractal dimension of an "active" time while the parameter (α) is related to the spatial fractal dimension in the percolating fractal system. In this way, the fractal dimension d_f and connectivity index (θ) of dynamical field's distribution is self-consistently related to the fractal dimension $\left(d_f^+\right)$ and the index of connectivity (θ^+) of fields.

4.5 The economical system self-organizing complexity

According to the experimental data analysis results presented previously and the theoretical framework of fractional dynamics the economical system is a globally hierarchical, self-similar and scale invariant physical system including nonlinear and non-local internal fractional dynamics, maintaining the hierarchical structure of the turbulence. Tsallis q-entropy principle included in his non-extensive statistical mechanics can reliably explain the economical system self-similar hierarchical turbulent structuring and phase transition processes presented in this study. According to [39], the Tsallis entropy principle can explain the spatial distribution of multifractal and intermittent economical system turbulent field.

The economical system, as any other system which lives far from equilibrium, can reveal metaequilibrium stationary states NESS as critical percolation states. These nonequilibrium states, similar to Boltzmann-Gibbs thermodynamical equilibrium states, can be produced as the system tends to obtain extremization of Tsallis q-entropy (S_q). The internal mechanism for this is the anomalous diffusion process in the physical space or the anomalous random walk in a hierarchical and multifractal structured phase space. The dynamics in the multifractal phase space is described by the fractional Langevin and the corresponding FFPK equations applied for variables of the economical system.

Moreover we conjecture that the metaequilibrium stationary states can be obtained also as the fixed points of a fractional renormalization flow equation in a fractal parameter space. This concept is the extension of the Chang's stochastic dynamics and renormalization group theory for space state [20]. Furthermore, the hierarchical, self-similar, multiscale and multifractal structure of the economical system at critical percolation and intermittent turbulent states can be obtained by the solution of the fractional Langevin equation and its corresponding FFBK equation, through N-point integral formulation. According to this theoretical modeling of economical system, we can estimate correlation functions related to the functional derivative of the q-partition function Z_q defined in the framework of non-extensive Tsallis statistical mechanics-thermodynamics [11]. In the following we present a more analytical sketch of these concepts.

4.6 Renormalization Group (RNG) theory and phase space transition

In this paragraph we present the renormalization theory of Chang which concerns with the multiscale and non-equilibrium behavior of non linear stochastic systems, which can be the mathematical basis for the understanding of economical crises as phenomena of phase transitions of the stochastic economical dynamics.

The results obtained by [25] justify the application of RNG theory for the description of the economical system scale invariance and the development of long – range correlation of the space intermittent turbulence state. Generally, and according to [20] the state space can be described by generalized Langevin stochastic equations of the general type:

$$\frac{\partial \varphi_i}{\partial t} = f_i\left(\vec{\varphi},\vec{x},t\right) + n_i\left(\vec{x},t\right) \qquad i = 1,2,... \tag{36}$$

Where f_i corresponds to the deterministic process concerning the dynamical variables $\phi(\vec{x},t)$ and n_i to the stochastic components (fluctuations). Generally, f_i are nonrandom forces corresponding to the functional derivative of the free energy functional of the system. According to [20] the behavior of a nonlinear stochastic system far from equilibrium can be described by the density functional P, defined by path integration of the system's stochastic Lagrangian

$$P\left(\vec{\varphi}\left(\vec{x},t\right)\right) = \int D\left(\vec{x}\right)\exp\left\{-i\cdot\int L\left(\dot{\vec{\varphi}},\vec{\varphi},\vec{x}\right)d\vec{x}\right\}dt \tag{37}$$

where $L\left(\dot{\varphi},\varphi,x\right)$ is the stochastic Lagrangian of the system, which describes the full dynamics of the stochastic system. Moreover, the far from equilibrium renormalization group theory applied to the stochastic Lagrangian L generates the singular points (fixed points) in the affine space of the stochastic distributed system. At fixed points the system reveals the character of criticality, as near criticality the correlations among the fluctuations of the random dynamic field are extremely long-ranged and there exist many correlation scales. Also, close to dynamic criticality certain linear combinations of the parameters, characterizing the stochastic Lagrangian of the system, correlate with each other in the form of power laws and the stochastic system can be described by a small number of relevant parameters characterizing the truncated system of equations with low or high dimensionality.

According to these theoretical results of Chang's theory, the stochastic economical system can exhibit low dimensional chaotic or high dimensional SOC like behavior, including fractal or multifractal structures with power law profiles. The power laws are connected to the near criticality phase transition process which creates spatial and temporal correlations as well as strong or weak reduction (self-organization) of the infinite dimensionality corresponding to a spatially distributed system. First and second phase transition processes can be related to discrete fixed points in the affine dynamical (Lagrangian) space of the stochastic dynamics. The SOC like behavior of dynamics corresponds to the second phase transition process as a high dimensional process at the edge of chaos. The process of strong and low dimensional chaos can be related to a first order phase transition process. The probabilistic solution of Eq. (36) of the generalized Langevin equations may include Gaussian or non-Gaussian processes as well as normal or anomalous diffusion processes depending upon the critical state of the system.

From this point of view, a SOC or low dimensional economical system, intermittent chaos or distinct q-statistical states with different values of the Tsallis q-triplet depends upon the type of the critical fixed (singular) point in the functional solution space of the system. When the stochastic system is externally driven or perturbed, it can be moved from a particular state of criticality to another characterized by a different fixed point and different dimensionality or scaling laws. Thus, the old SOC theory could be a special kind of critical dynamics of an externally driven stochastic system. After all, in an economical system, SOC and low dimensional chaos can coexist in the same dynamical system as a process manifested by different kinds of fixed (critical) points in its solution space. Due to this fact, the economical system dynamics may include high dimensional SOC process or low dimensional chaos or other more general dynamical process corresponding to various *q*-statistical states.

The non-extensive character of the economical system related to *q*-statistical metaequilibrium thermodynamics as well as the existence of long-range correlations has to be harmonized with the nonlinear dynamics of the economical system. The experimental results of this study press us to look for a dynamical mechanism efficient to explain the spontaneous development of the long-range correlations and reduction of the infinite degrees of freedom corresponding to the distributed character of the economical system. This is the base for the development of anomalous diffusion processes, long-range correlations and scale invariance which can be amplified as the system approaches far from equilibrium dynamical critical points. Also, it is known that nonlinear dissipative dynamics with finite or infinite degrees of freedom includes the possibility of self-organizing reduction of the effective degrees of freedom and bifurcation to periodic or strange (chaotic) attractors with spontaneous development of macroscopic ordered spatiotemporal patterns. The bifurcation points of the nonlinear dynamics, corresponds to the critical points of far from equilibrium non-classical statistical mechanics and its generalizations, as well as to the fixed points of the renormalization group theory (RGT). The RGT is based in the general principle of scale invariance of the physical processes as we pass from the microscopic statistical continuum limit to the macroscopic thermodynamic limit.

According to [41] the dynamics of system as it approaches the critical points includes a cooperation of all the scales from the microscopic to macroscopic level. The multiscale and holistic dynamics can be produced by scale invariance principle included in the RG transformation corresponding to the flow:

$$\vec{K}^n = R_l\left(\vec{K}^{n-1}\right) = \ldots = R_l^n \cdot \left(\vec{K}^0\right) \qquad n = 0,1,2\ldots \quad , \tag{38}$$

where $\vec{K}^0$ is the original parameter vector $\vec{K}^0$ and R_l is the renormalization-group operator of the partition function and the density of free energy. The parameter vector $\vec{K}$ has as components the parameters $\vec{K}_1, \vec{K}_2, \ldots, \vec{K}_m$ the coupling constants upon which the free energy depends. The RG flow in the dynamical parameter space of vectors $\vec{K}$ is caused by a spatial change of scale as at every step of flow in the parameter space the spatial scale is rescaled according to the relation: $\vec{r}' = l^{-1} \cdot \vec{r}$. As the free energy and the partition function are rescaled by the RG flow in the parameter space the correlation length (ξ) is rescaled also according to the relation:

$$\xi\left(\vec{K}^n\right) = l^{-1} \cdot \xi\left(\vec{K}^{n-1}\right) = \ldots = l^{-n} \cdot \xi\left(\vec{K}^0\right) \tag{39}$$

At the fixed points $\vec{K}^*$ of the RG flow, the relation:

$$\xi\left(\vec{K}^*\right) = l^{-1} \cdot \xi\left(\vec{K}^*\right) \tag{40}$$

implies that the correlation length at the fixed point must be either zero or infinite. Also, as the zero value is without physical interest we conclude the infinite correlation of the system at the fixed point $\vec{K}^*$ or long-range correlation in near the fixed point. The dynamics of the system near the physical critical point corresponds to the flow of the parameter vector $\vec{K}$ at the neighborhood of the fixed point. The flow of the parameter vector $\vec{K}$ at the neighborhood of the fixed point $\vec{K}^*$ is a nonlinear flow in a finite dimensional space which survives the most significant physical characteristics of the original dynamics of the economical system with infinite degrees of freedom. The representation of the infinite dimensional dynamics to a finite dimensional is possible at every instant the infinite dimensional dynamical state (state of infinite degrees of freedom) is transformed by the scale invariance vehicle to a finite dimensional dynamics in the parameter space. According to this theoretical description the economical system can exist at district fixed points in the parameter points corresponding to the and active states. From the above theoretical point of view the quantative change of the non-extensive Tsallis statistical characteristics corresponds to the economical system's nonequilibrium fixed points variations related to the RGT. The change of the dynamical RGT fixed points can be identified as topological phase transition process.

5. Conclusions

In this paper we present various aspects of the general theory of Complexity, developed also for physical complex distributed systems, and a possible way of how they can be applied to the dynamics of economical systems. Nonlinear time series analysis of economical time series justifies this effort, which is further supported by the universality of the theory of Complexity. For the interpretation of the results of the nonlinear time series analysis of economical time series we need to extend the theoretical framework of economical complexity into various ideas such as strange dynamics, topological phase transition ,anomalous diffusion, fractal - multifractal topology and fractional dynamics ,renormalization and scale invariance ,etc.

In particular, the estimation of Tsallis *q*-triplet for the two stock time series showed that an economical system can have characteristics such as non-extensivity and long range correlations, intermittent turbulence, multifractal - multiscale and non-equilibrium phase transitions, just like in physical complex systems. This experimental evidence reveals a need for a new theoretical description of economical dynamics, just like the one we described on previous paragraphs. Therefore, new theoretical concepts such as strange and non-Gausian dynamics, fractal topology and fractional dynamics, Tsallis entropy, non extensive dynamics and non equilibrium phase transition processes, are some of the most closely related theoretical concepts to the complex character of economical system developed in this study.

Summarizing, Tsallis q-entropy principle can reliably explain the economical system self-similar hierarchical turbulent structuring and phase transition processes presented in this study. The economical system, as any other system which lives far from equilibrium, can reveal metaequilibrium stationary states (NESS) as critical percolation states. These nonequilibrium states, similar to Boltzmann-Gibbs thermodynamical equilibrium states, can be produced as the system tends to obtain extremization of Tsallis q-entropy (S_q). The quantitative change of the non-extensive Tsallis statistics of the economical system's system can be related to the renormalization group theory (RGT) change of the fixed points (NESS) in the dynamical parameter space of the economical system dynamics. The internal mechanism for this is the anomalous diffusion process in the physical space or the anomalous random walk in a hierarchical and multifractal structured phase space. The dynamics in the multifractal phase or physical space is described by the fractional equations (e.g Langevin and the corresponding FFPK equations). Moreover, we conjecture that the metaequilibrium stationary states can be obtained also as the fixed points of a fractional renormalization flow equation in a fractal parameter space. Also, the hierarchical, self-similar, multiscale and multifractal structure of the economical system at critical percolation and intermittent turbulent states can be described by the solution of the fractional Langevin equation, as the N-point correlation functions related to the functional derivative of the q-partition function Z_q defined in the framework of non-extensive Tsallis statistical mechanics-thermodynamics.

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