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Chaotic Effect of Linear Marginal Cost in Nonlinear Duopoly Game with Heterogeneous Players

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Abstract

In this study we investigate the dynamics of a nonlinear discrete-time duopoly game, where the players have heterogeneous expectations linear demand and quadratic cost functions. Two players with different expectations are considered; one is boundedly rational and the other thinks with adaptive expectations. We show that the model gives more complex chaotic and unpredictable trajectories as a consequence of change in the slope of the marginal costs. The chaotic features are justified numerically via computing Lyapunov numbers, sensitive dependence on initial conditions and the box dimension of the chaotic attractor.

Keywords: Discrete dynamical system; Cournot duopoly games; Complex dynamics; Heterogeneous expectations; Box dimension

1. Introduction

An Oligopoly is a market structure between monopoly and perfect competition, where there are only a few number of firms in the market producing homogeneous products. The dynamic of an oligopoly game is more complex because firms must consider not only the behaviors of the consumers, but also the reactions of the competitors i.e. they form expectations concerning how their rivals will act.

Cournot, in 1838 [9] has introduced the first formal theory of oligopoly. He treated the case with naive expectations, so that in every step each player (firm) assumes the last values that were taken by the competitors without estimation of their future reactions. Expectations play an important role in modelling economic phenomena. A producer can choose his expectations rules of many available techniques to adjust his production outputs. Some authors considered duopolies with homogeneous expectations and found a variety of complex dynamics in their games, such as appearance of strange attractors [1,2,5,14,19,21]. Also models with heterogeneous agents were studied [3,4,11,22]. In this paper we study the dynamics of a duopoly model where each firm behaves with different expectations strategies, so we are going to apply this kind of beliefs to a duopoly game. This kind of beliefs is common in real world problems such as economic, biology and social sciences problems. We consider a duopoly model where each player forms a different strategy in order to compute his expected output. We take firm 1 to represent a boundedly rational player while firm 2 has adaptive expectations. Each player adjusts his outputs towards the profit maximizing amount as target by using his expectations rule.

The main purpose of this paper is to investigate the effect of the slope of the marginal cost in the dynamic behavior of the duopoly examined in Dubiel- Teleszynski [10]; representing two firms using heterogeneous expectations rules, linear inverse demand and quadratic cost functions. The plan of the paper is as follows: In Section 2, the dynamics of a duopoly game with boundedly rational player and adaptive player is analyzed. The existence, local stability and bifurcation of the equilibrium points are also analyzed. In Section 3 numerical simulations are used to show complex dynamic via computing Lyapunov numbers, sensitive dependence on initial conditions and the box dimension of the chaotic attractor is calculated.

2. The model

We consider a simple Cournot-type duopoly market where firms (players) produce homogeneous goods which are perfect substitutes and offer them at discrete-time periods $t = 0,1,2,...$ on a common market. At each period t, every firm must form an expectation of the rival's output in the next time period in order to determine the corresponding profit-maximizing quantities for period $t+1$.

The inverse demand function of the duopoly market is assumed linear and decreasing:

$$P = f(Q) = a - b(q_1 + q_2) \quad (1)$$

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where $Q = q_1 + q_2$ is the industry output and $a,b > 0$. It is supplied by two firms with quadratic cost function

$$C_i(q_i) = c_i q_i^2,\ \text{i=1,2} \tag{2}$$

With these assumptions the single profit of ith firm is given by

$$\Pi_i(q_1,q_2) = q_i(a - bQ) - c_i q_i^2,\ \text{i=1,2} \tag{3}$$

Then the marginal profit of ith firm at the point (q_1,q_2) of the strategy space is given by

$$\frac{\partial \Pi_i}{\partial q_i} = a - 2(b + c_i)q_i - bq_j,\ \text{i,j=1,2,i} \neq \text{j} \tag{4}$$

This optimization problem has unique solution in the form

$$q_i = g(q_j) = \frac{1}{2(b + c_i)}\left(a - bq_j\right) \tag{5}$$

2.1. Duopoly game with heterogeneous players
The first firm decides to increase its production if it has a positive marginal profit, or decreases its production if the marginal profit is negative (boundedly rational player). Then the dynamical equation of player 1 has the form

$$q_1(t+1) = q_1(t) + uq_1(t)\frac{\partial \Pi_i}{\partial q_1},\ \text{t=0,1,2,...} \tag{6}$$

where u is a positive parameter which represents the relative speed of adjustment. Another expectation rule that firm can use to revise their beliefs according to the adaptive expectations rules. If the firm 2 think with adaptive expectations it computes its outputs with weights between last period's outputs and his reaction function $g(q_1)$. Hence the dynamic equation of the adaptive expectation player 2 has the form,

$$q_2(t+1) = (1 - v)q_2(t) + vg(q_2(t)) \tag{7}$$

Where $v \in [0,1]$ is a speed of adjustment of adaptive player. Hence the dynamic duopoly game in this case is formed from combining Eqs. (6) and (7). Then the dynamical system of heterogenous players is described by

$$\begin{cases} q_1(t+1) = q_1(t) + uq_1(t)\left(a - 2(b + c_1)q_1 - bq_2\right) \\ q_2(t+1) = (1 - v)q_2(t) + \dfrac{v}{2(b + c_2)}\left(a - bq_1(t)\right) \end{cases} \tag{8}$$

We will focus on the dynamics of the system (8) to the parameters $c_i, i = 1,2$.

2.2. Equilibria and local stability

The equilibria of the dynamical system (8) are obtained as nonnegative solutions of the algebraic system

$$\begin{cases} q_1(t)\left(a - 2(b + c_1)q_i - bq_2\right) = 0 \\ \dfrac{1}{2(b + c_2)}\left(a - bq_1(t)\right) - q_2(t) = 0 \end{cases} \tag{9}$$

Which obtained by setting $q_i(t+1) = q_i(t),\ i = 1,2$ in Eq. (8) and we can have at most two equilibriums $E_0 = \left(0, a/2(b + c_2)\right)$ and $E^* = (q_1^*, q_2^*)$.

The fixed point E_0 is called boundary equilibrium. The second equilibrium E^* is called Nash equilibrium where

$$\begin{aligned} q_1^* &= \frac{a(b + 2c_2)}{3b^2 + 4b(c_1 + c_2) + 4c_1c_2} \\ q_2^* &= \frac{a(b + 2c_1)}{3b^2 + 4b(c_1 + c_2) + 4c_1c_2} \end{aligned} \tag{10}$$

The study of the local stability of equilibrium solutions is based on the localization on the complex plane of the eigenvalues of the Jacobian matrix of the two dimensional map (Eq. (12)). In order study the local stability of equilibrium points of the model (8), we consider the Jacobian matrix along the variable strategy (q_1,q_2). According to [10] E_0 is saddle point of the system (8) and the Nash equilibrium E^* is locally stable if the following conditions are hold

$$\begin{cases} (i)\ \ 1 - T + D > 0 \\ (ii)\ \ 1 + T + D > 0 \\ (iii)\ \ 1 - D > 0 \end{cases} \tag{11}$$

where T is the trace and D is the determinant of the Jacobian matrix

$$J(E^*) = \begin{bmatrix} 1 - 2u(b + c_1)q_1^* & -ubq_1^* \\ \dfrac{-vb}{2(b + c_2)} & 1 - v \end{bmatrix} \tag{12}$$

Conditions (i) and (iii) is always satisfied, whereas condition (ii) define an unbounded region of stability in the parameters space (u,v).Then the second condition is the condition for the local stability of Nash equilibrium which becomes:

$$4 - 2v - 4u(b + c_1)q_1^* + \frac{uva(b + 2c_2)}{2(b + c_2)} > 0 \tag{13}$$

3. Numerical simulations

To provide some numerical evidence for the chaotic behavior of the system Eq. (8), as a consequence of change in the slope of the marginal costs of the players, we present various numerical results here to show the chaoticity, including its bifurcations diagrams, strange attractors, Lyapunov numbers and sensitive dependence on initial conditions [15]. In order to study the local stability properties of the equilibrium points, it is convenient to take the parameters values as follows: $a=10, b=0.5$ Numerical experiments are computed to show the bifurcation diagram with respect to c_1, c_2, strange attractor of the system (8) in the phase plane of the quantity outputs (q_1, q_2), the Lyapunov numbers and the box dimension of the chaotic attractor. Figs. 1 shows the bifurcation diagrams with respect to the parameter c_1 and for u=0.3, v=0.6, c_2= 0.8. Figs. 2 show the bifurcation diagrams with respect to the parameter c_2 and for u=0.35, v=0.6, c_1= 0.6. In these figures for very small values of the parameter c_i one observes complex dynamic behavior such as cycles of higher order and chaos. Figs. 3, 4 shows the graph of strange attractor and Lyapunov numbers of the orbit of (0.1, 0.1) for u= 0.35, v=0.5, c_1 =1, c_2 =0.4. From these results when all parameters are fixed and only c_i is varied the structure of the market of duopoly game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.

3.3.1. Sensitive dependence on initial conditions

To demonstrate the sensitivity to initial conditions of the system (8), we compute two orbits with initial points (0.1, 0.2) and (0.1, 0.2001), respectively and with the parameters values u=0.35, v=0.5, c_1= 0.7, c_2= 0.8. The results are shown in Figs. 5. At the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly. From Figs. 5 we show that the time series of the system Eq. (8) is sensitive dependence to initial conditions, i.e. complex dynamics behaviors occur in this model.

3.3.2. Box Dimension

One way to measure the complexity of a set (an orbit of the map) is to compute its dimension over different scales of magnification [15]. Let S a bounded set in $\mathbb{R}^m$ and $N(r)$ the minimum number of boxes of side-length r needed to contain all the points of the set. The box dimension BD(S) of S is defined to be the number d that satisfies:

$$N(r) = \lim_{r\to 0} k r^{-d} \tag{14}$$

Where k is proportionality constant. In practice, we find d by taking the logarithm of both sides of Eq. (14) (before taking the limit) to find

$$d = BD(S) = -\lim_{r\to 0}\frac{\ln N(r)}{\ln r}$$

(15)

When the limit exists. In most cases, the only practical way of calculating the box dimension of an orbit is through numerical approximation. If S is the orbit of Fig. 5 and $r = 0.001$, $BD(S) = 1.09368$

3.3.3. Figures

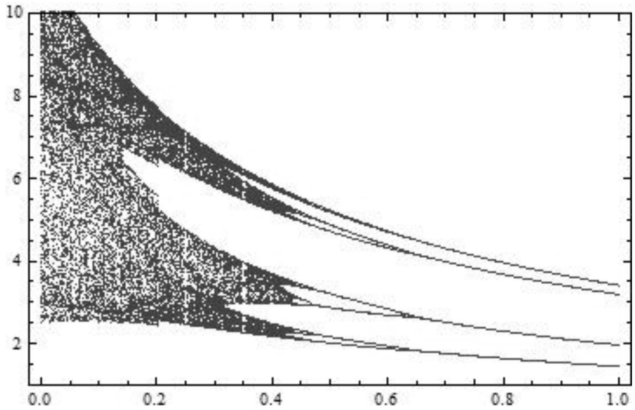

Fig.1(a). Bifurcation diagram with respect to the parameter c_1 against variable $q_1$and $c_1 \in [0,11]$ (left), $c_1 \in [0,1]$ for u=0.3, v=0.6, c_2= 0.8 and 550 iterations of the map Eq. (8).

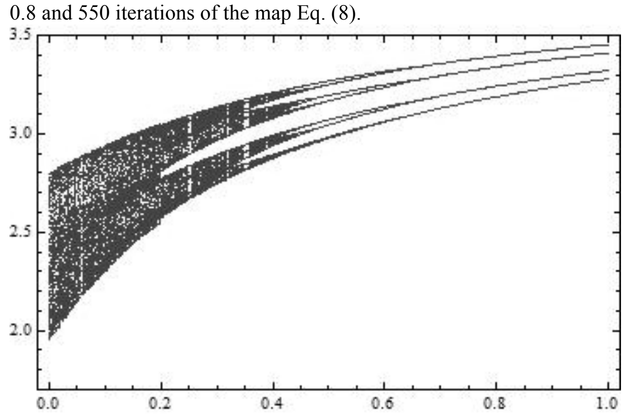

Fig.1(b). Bifurcation diagram with respect to the parameter c_1 against variable q_2 and $c_1 \in [0,1]$ for u=0.3,v=0.6, c_2= 0.8 and 550 iterations.

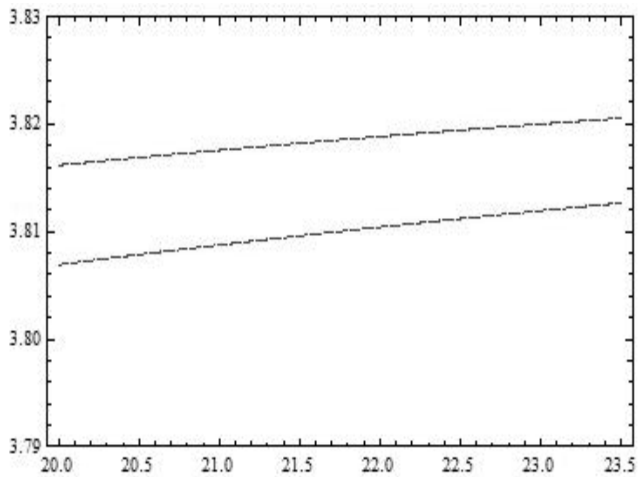

Fig. 1 (c). Bifurcation diagram with respect to the parameter c_1 against variable q_2 and $c_1 \in [20,23.5]$ for u=0.3,v=0.6, c_2= 0.8 and 550 iterations.

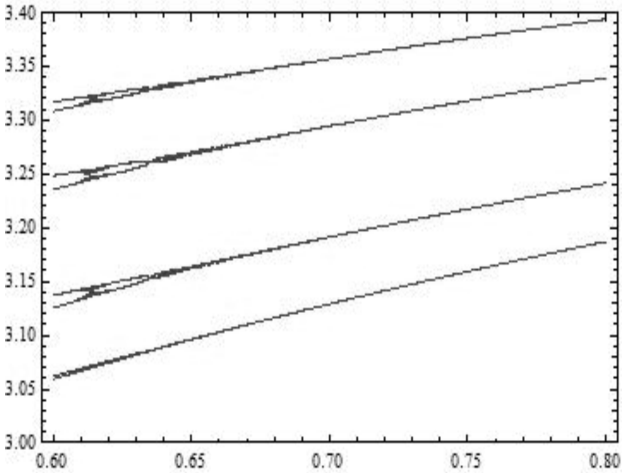

Fig.1(d). Bifurcation diagrams with respect to the parameter c_1 against variable q_2 and $c_1 \in [0.6,0.8]$ for u=0.3, v=0.6, c_2= 0.8 and 550 iterations.

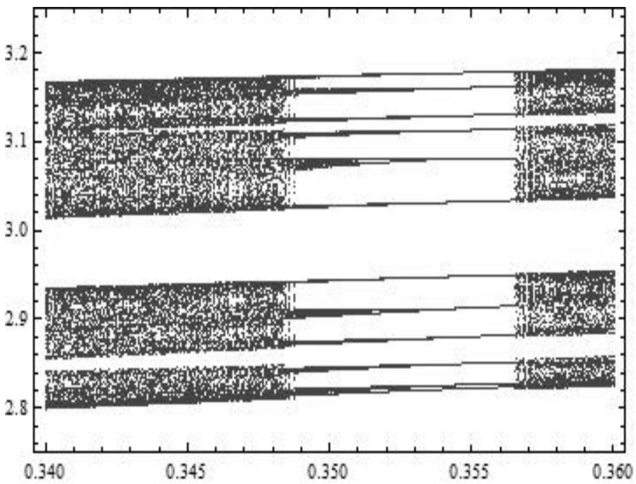

Fig.1(e). Bifurcation diagrams with respect to the parameter c_1 against variable q_2 and $c_1 \in [0.34,0.36]$ for u=0.3, v=0.6, c_2= 0.8 and 550 iterations.

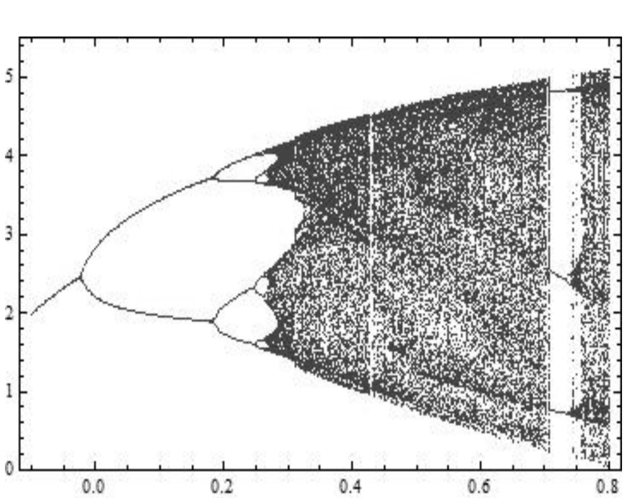

Fig 2 (a). Bifurcation diagram with respect to the parameter c_2 against variable q_1 and $c_2 \in [-0.1,0.8]$ for u=0.35,v=0.6, c_1= 0.6 and 550 iterations.

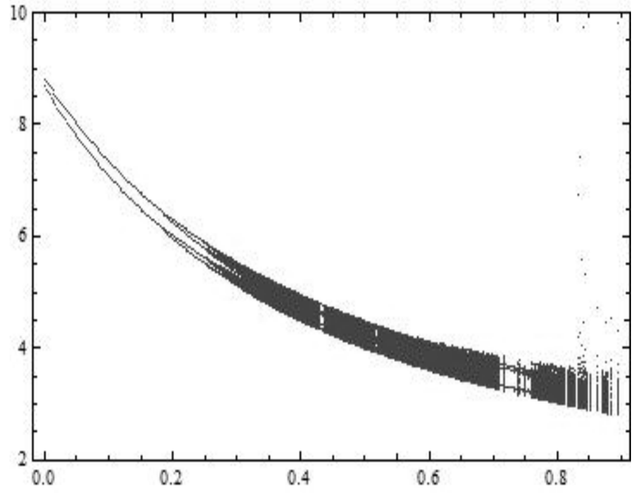

Fig.2(b). Bifurcation diagram with respect to the parameter c_2 against variable q_2 and $c_2 \in [0,1]$ for u=0.35, v=0.6, c_1= 0.6 and 550 iterations.

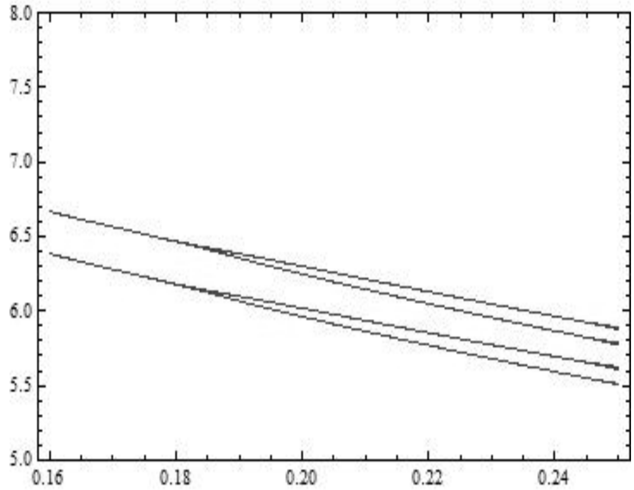

Fig.2(c). Bifurcation diagram with respect to the parameter c_2 against variable q_2 and $c_2 \in [0.16,0.25]$ for u=0.35,v=0.6, c_1= 0.6 and 550 iterations.

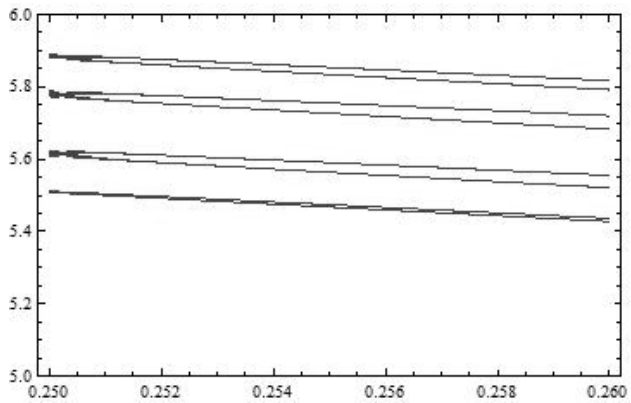

Fig.2(d). Bifurcation diagrams with respect to the parameter c_2 against variable q_2 and $c_2 \in [0.25,0.26]$ for u=0.35,v=0.6, c_1= 0.6 and 550 iterations.

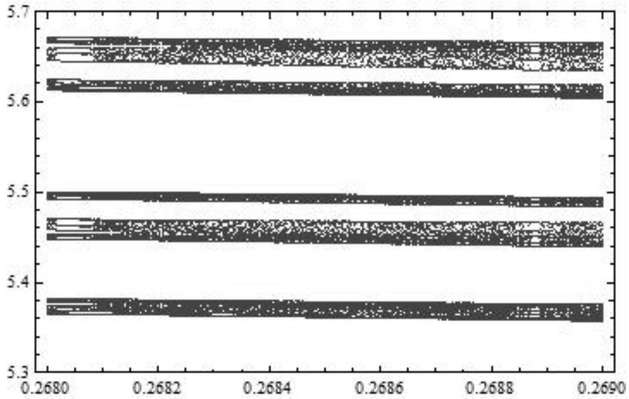

Fig.2 (e). Bifurcation diagrams with respect to the parameter c_2 against variable q_2, $c_2 \in [0.268, 0.269]$ for u=0.35,v=0.6, c_1= 0.6 and 550 iterations.

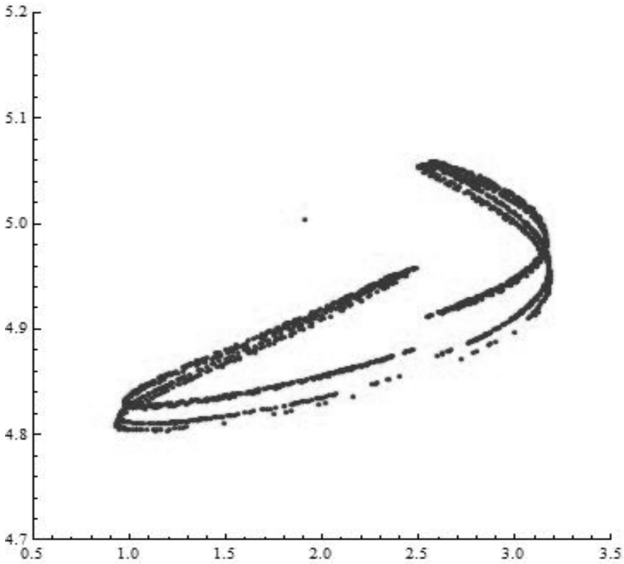

Fig.3. Strange atractor with 2000 iterations of the map and initial point (0.1, 0.1), for u= 0.35, v=0.5, c_1 =1, c_2 =0.4.

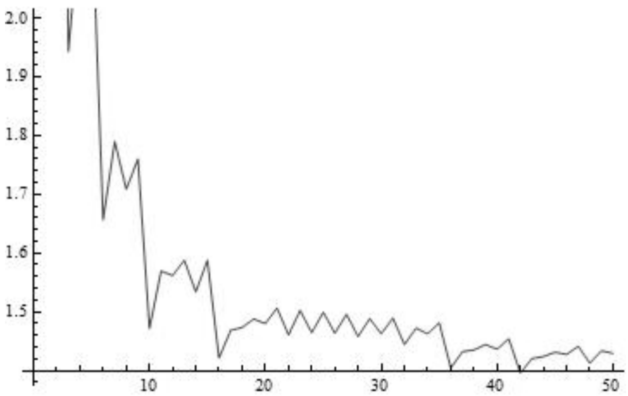

Fig.4. Lyapunov numbers versus the number of iterations of the orbit orb. (0.1, 0.1), for u= 0.35, v=0.5, c_1 =1, c_2 =0.4.

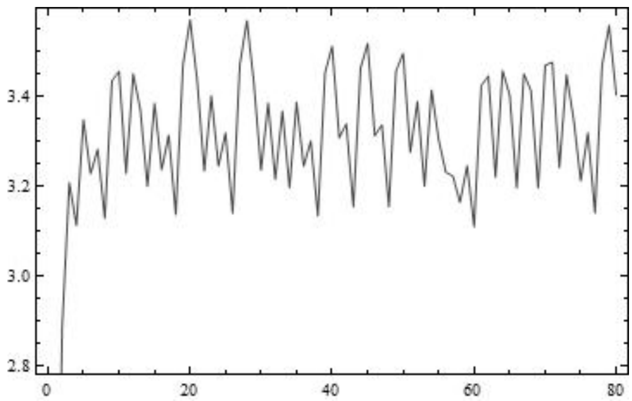

Fig.5 (a). Sensitive dependence on initial conditions for q_2 –coordinate plotted against the time: The orbit of the point (0.1, 0.2) for the system (8), with the parameters values u=0.35, v=0.5, c_1= 0.7, c_2= 0.8.

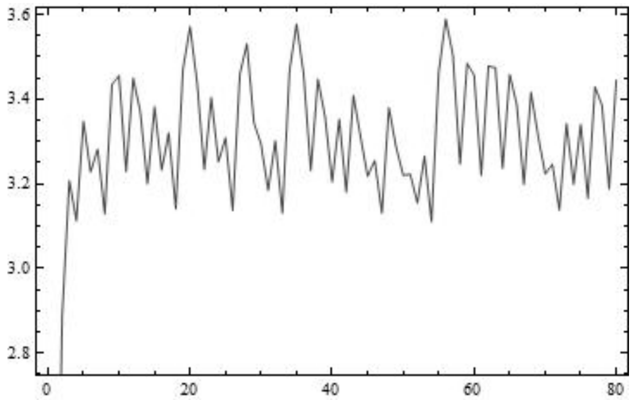

Fig.5 (b). Sensitive dependence on initial conditions for q_2 –coordinate plotted against the time: The orbit of the point (0.1, 0.2001) for the system (8), with the parameters values u=0.35, v=0.5, c_1= 0.7, c_2= 0.8.

4. Conclusion

In this paper we proposed and analyzed the effect of the slope of a linear marginal cost in the dynamic behavior of a nonlinear duopoly game, which contains two-types of heterogeneous players: boundedly rational player and adaptive expectation player. We show that this parameter may change the stability of the system and cause a structure to behave chaotically. For very small values of this parameter the dynamical system becomes unstable, through period-doubling bifurcation.

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