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Research Article

Dynamical Properties and Chaos Synchronization in a Fractional-Order Two-Stage Colpitts Oscillator

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Abstract

In this paper, the dynamics and synchronization of a fractional-order four dimensional nonlinear system based on a two-stage Colpitts oscillator is investigated, using the Grünwald-Letnikov method. The study of the fractional-order stability of the equilibrium states of the system is carried out. The bifurcation diagram confirms the occurrence of Hopf bifurcation in the proposed system when the fractional-order passes a sequence of critical values, and reveals in addition various bifurcation scenarios including period-doubling and interior crisis transitions to chaos. In order to promote chaos-based fractional-order synchronization of this type of oscillators, a synchronization strategy based upon the design of a nonlinear state observer is successfully adapted. Numerical simulations are performed to demonstrate the effectiveness and applicability of the proposed technique.

Keywords: Colpitts Oscillator, Fractional-order, Synchronization, Hopf Bifurcation, Nonlinear systems, Chaos theory.

1. Introduction

Fractional calculus has an about 300-year-old history, but its applications to physics and engineering is rather recent [1]. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [2], dielectric polarization [3], electromagnetic waves [4], and electromechanical systems [5] just to name some.

Since some decades, there is a growing interest in investigating the chaotic behavior and dynamics of fractional-order dynamic systems; this can be understood as it has been found that fractional-order systems possess memory and display more sophisticated dynamics compared to its integral-order counterparts, something that is of great significance in secure communication [6-24]. It has been shown that several chaotic systems can remain chaotic when their models become fractional [5, 6]. In reference [7], using the predictor-corrector scheme, the authors proposed the bifurcation of fractional-order diffusionless Lorenz System. It was also shown that the fractional-order Qi oscillator with order as low as 0.96 can produce a chaotic attractor [8]. A fractional variational optical flow model is introduced in [9], and a new class of nondiffracting fractional vortex beams that connect Bessel beams of successive order in a smooth transition is introduced by [10]. A three-dimensional fractional-order modified hybrid optical system is presented in [11] where it was shown that Hopf bifurcation occurs on the proposed system when the fractional-order varies and passes a sequence of critical values. The growing interest for fractional-order differential equations is also sustained by recent developments in the area of Mathematics [12].

On the other hand, in the past two decades, a new direction of chaos research has emerged to address the more challenging problem of chaos synchronization due to its potential applications in laser physics, chemical reactions, secure communication, biomedicine and so on [25-27]. The thrust of research within this area aims at achieving master-slave synchronization between two chaotic systems by choosing various kinds of methods following the pioneering work of Pecora and Carroll [28]. In reference [14], an adaptive feedback control scheme for the synchronization of two coupled chaotic fractional-order systems with different fractional orders has been proposed. Based on the fractional Routh–Hurwitz conditions and using specific choice of linear feedback controllers, Zhang, Wang and Fang [15]

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showed that the Newton–Leipnik system is controlled to its equilibrium points. Furthermore, Lu has proposed a nonlinear observer to synchronize a class of identical fractional-order chaotic systems [16]. In reference [8] the chaos synchronization problem of the fractional-order Qi oscillators coupled in master-slave pattern is examined by applying three different kinds of methods: the nonlinear feedback method, the one-way coupling method and the method based on the state observer. The active control method is the plinth on which the work in Ref. [13] is based. Despite these many examples the bifurcation of fractional-order nonlinear system has been studied using solely the Caputo derivative definition, except for [24] and are generally limited to three dimensional systems.

In the present work, we propose to tackle the problem of bifurcation of a four dimensional fractional-order nonlinear system. The two-stage well studied Colpitts oscillator presented in reference [29] offers a good candidate for the study, due to its broad band in frequency domain. The Grünwald-Letnikov fractional derivative defined in [17] will be used instead of the Caputo, for we found it more adequate for our study. We propose a nonlinear feedback controller for the achievement of synchronization of two identical two-stage Colpitts oscillators. On the basis of fractional-order Lyapunov stability theory we propose a feedback gains controller leading to the synchronization. Numerical simulations demonstrate the applicability and efficiency of the nonlinear control law and verify the theoretical results of the paper.

The rest of this paper is organized as follows: In Section 2, the fractional-order system developed around a two-stage Colpitts oscillator is proposed and its dynamics studied. The numerical results of the dynamics are presented and discussed in Section 3, while the next Section is devoted to the synchronization of two two-stage Colpitts oscillators. Finally, Section 5 concludes the work.

2. The fractional-order of a Two-stage Colpitts Oscillator

2.1 Basic definition and preliminaries

To discuss fractional-order chaotic systems, we often need to solve fractional-order differential equations. For analytic calculation of fractional-order derivatives, we use two theorems.

Theorem 1 [18, 19]

The following commensurate order system:

$$ {}_{O}^{C}D_t^q x(t) = Ax(t),\ x(0) = x_0, \quad (1)$$

With $0 \le q \le 1$, $x \in M^n$ and $A \in M^{nxn}$ is asymptotically stable if and only if $|\arg(\lambda)| > q\frac{\pi}{2}$ is satisfied for all eigenvalues λ of the matrix A. Moreover, this system is stable if and only if $|\arg(\lambda)| \ge q\frac{\pi}{2}$ is satisfied for all eigenvalues λ of A with those critical eigenvalues that satisfy $|\arg(\lambda)| = q\frac{\pi}{2}$ having geometric multiplicity of one.

Theorem 2 [20]

Consider the following linear fractional order system:

$$ {}_{O}^{C}D_t^q x(t) = Ax(t),\ \text{with}\ x(0) = x_0 \quad (2)$$

where $x \in M^n$, $A \in M^{nxn}$ and $q = (q_1\ q_2\ ...\ q_n)^T$, with $0 < q_i \le 1$ and $q_i = \frac{n_i}{d_i}$, $\gcd(n_i, d_i) = 1$. Let M be the lowest common multiple of the denominators d_i's. The zero solution of Eq.3 is globally asymptotically stable in the Lyapunov sense if all roots λ's of the equation:

$$ \Delta(\lambda) = \det(diag(\lambda^{Mq_i}) - A) = 0 \quad (3)$$

satisfy $|\arg(\lambda)| > \frac{\pi}{2M}$.

For numerical calculation of fractional-order derivatives, there are three commonly used definitions. The Grünwald-Letnikov (GL) method [17, 30] is given in the following Eq.4:

$$ {}_aD_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-\alpha}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh), \quad (4)$$

where [.] indicates the integer part.

The Riemann-Liouville (RL) definition follows as:

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \ \text{for}\ (n-1 < a < \text{n}) \quad (5)$$

where $\Gamma(.)$ is the *gamma* function. The Caputo definition of fractional derivatives can also be recalled as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,\ \text{for}\ (n-1 < a < \text{n}) \quad (6)$$

Based on the fact that for a wide class of functions the three definitions - GL (4), RL (5), and Caputo's (6) - are equivalent if $f(a) = 0$, we can then use the relation (7) derived from the GL definition (4). The new relation for the explicit numerical approximation of q-th derivative at the points kh, $(k = 1, 2, ...)$ has the following form:

$$ {}_{(k-L_m/h)}D_{t_k}^q f(t) \approx h^{-q} \sum_{j=0}^{h} (-1)^j \binom{q}{j} f(t_k - j) = h^{-q} \sum_{j=0}^{h} C_j^{(q)} f(t_k - j) \quad (7)$$

where L is the "memory length", $t_k = kh$, with h the time step of calculation and $C_j^{(q)} (j = 0, 1, ...k)$ the binomial coefficients. For their calculation we can use for instance the following expression:

$$ C_0^{(q)} = 1,\ C_j^{(q)} = \left(1 - \frac{1+q}{j}\right) C_{j-1}^{(q)}. \quad (8)$$

The binomial coefficients $C_j^{(q)} (j = 0, 1, ...k)$ can also be expressed using a factorial. The gamma function $\Gamma(n) = (n-1)!$ can allow the generalization of the binomial coefficient to non-integer argument. Thus, relation (8) can be rewritten as follows [30]:

$$(-1)^j \binom{q}{j} = (-1)^j \frac{\Gamma(q+1)}{\Gamma(1+1)\Gamma(q-j+1)} = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} \quad (9)$$

2.2 Dynamics of the system

The proposed four dimensional fractional-order system under study described by the set of Eq.10 is obtained by modifying the integer-order two-stage Colpitts oscillator proposed in Ref. [29]:

$$\begin{cases} D^{q_1} x_1 = \sigma_1 \left(x_4 - \gamma\phi\left(x_2 + x_3 \right) \right) \\ D^{q_2} x_2 = x_4 \\ D^{q_3} x_3 = \sigma_2 \left(x_4 - \gamma\phi\left(x_2 \right) \right) \\ D^{q_4} x_4 = -x_1 - x_2 - x_3 - \varepsilon x_4. \end{cases} \quad (10)$$

Here, the parameters σ_1, σ_2, γ and ε are positive real's, $\phi(y) = \exp(-y) - 1$ and $q = (q_1, q_2, q_3, q_4)$ is the fractional-order. According to Ref. [29], when $q = (1,1,1,1)$, the set of Eq.10 exhibits chaotic behavior with the parameter values $\sigma_1 = 1.25,\ \sigma_2 = 1,\ \gamma = 1.5385,$ and $\varepsilon = 1.175$.

2.2.1 Stability of the equilibrium points

In this section we proceed with commensurate order $q = q_1 = q_2 = q_3 = q_4$ [18, 19]. Fractional-order of the proposed two-stage Colpitts oscillator (10), when $(\sigma_1,\ \sigma_2,\ \gamma,\ \varepsilon) = (1.25,\ 1,\ 1.9,\ 1.175)$ has one equilibrium point, $O = (0,\ 0,\ 0,\ 0)$. The Jacobian matrix of system described by the set of Eq.10, evaluated at the equilibrium point is:

$$J_{|O} = \begin{pmatrix} 0 & \sigma_1\gamma & \sigma_1\gamma & \sigma_1 \\ 0 & 0 & 0 & 1 \\ 0 & \sigma_2\gamma & 0 & \sigma_2 \\ -1 & -1 & -1 & -\varepsilon \end{pmatrix}. \quad (11)$$

For $q = (0.96,\ 0.96,\ 0.96,\ 0.96)$ around the equilibrium point O, the equation $\det(diag(\lambda^{Mq_i}) - J_{|O}) = 0$ with $i = 1,2,3,4$ and M = 100becomes,

$$\lambda^{384} + \varepsilon\lambda^{288} + (1+\sigma_1+\sigma_2)\lambda^{192} + (\sigma_1\sigma_2 + \sigma_1 + \sigma_2)\gamma\lambda^{96} + \sigma_1\sigma_2\gamma^2 = 0. \quad (12)$$

Thus, for $\gamma = 1.1$, $\min_i \left\{ \left| \arg(\lambda) \right| \right\} = 0.01596 > \frac{\pi}{200}$ and for $\gamma = 1.2$, $\min_i \left\{ \left| \arg(\lambda) \right| \right\} = 0.01565 < \frac{\pi}{200}$.

Therefore, on the basis of [17], system named Eq.10 is asymptotically stable at equilibrium point O for $\gamma = 1.1$, and unstable for $\gamma = 1.2$.

2.3 Hopf bifurcation

One of the basic differences between the dynamical behavior of fractional-order systems and that of integer-order systems is that the limit set of a trajectory of integer-order system such as a limit cycle is solution for the system under consideration, while in the case of fractional-order systems, such a limit set of a trajectory may not be solution for this system [21]. In reference [22], the authors claimed that there are no periodic orbits in fractional order systems, and in [23], an example is given where the solutions of the system are also not periodic, but do converge to periodic signals, confirming in both cases what has been stipulated in [21].

In the present paper, we consider the final state of the trajectory that appears at the Hopf bifurcation (after suppression of the transitory state). It is also not a periodic solution of the fractional-order system given by Eq.10, but attracts nearby solutions.

Let us consider the following four-dimensional fractional-order commensurate system:

$$D^q x = f(\gamma, x), \quad (13)$$

Where $q \in]0,2[$, $x \in \mathbb{R}^4$ and suppose that E is an equilibrium point of this system. In the integer case ($q = 1$), the stability of E is related to the sign of $\mathrm{Re}(\lambda_i)$, $i = 1,2,3,4$ where λ_i are the eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial x}|_E$. If $\mathrm{Re}(\lambda_i) < 0$ for all $i = 1,2,3,4$ then E is locally asymptotically stable. If there exist an i for which $\mathrm{Re}(\lambda_i) > 0$, then E is unstable.

To undergo a Hopf bifurcation at the equilibrium point E when $\gamma = \gamma^*$, Eq.10 with q = 1 must fulfill the following conditions:

- The Jacobian matrix must have two pairs of complex-conjugate eigenvalues $\lambda_{1,2}(\gamma) = \theta_1(\gamma) \pm i\eta_1(\gamma)$, and $\lambda_{3,4}(\gamma) = \theta_2(\gamma) \pm i\eta_2(\gamma)$.
- $\theta_j(\gamma^*) = 0$, with $j = 1,2$,
- $\eta_j(\gamma^*) \neq 0$, with $j = 1,2$, and finally
- $\frac{\partial \theta_j}{\partial \gamma}|_{\gamma=\gamma^*} \neq 0$.

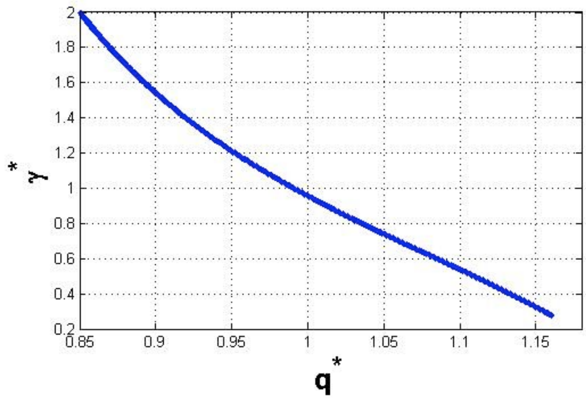
In the fractional case, the stability of E is related to the sign of $m_i(q,\gamma) = q\frac{\pi}{2} - \left| \arg(\lambda_i(\gamma)) \right|$, with $i = 1,\ 2,\ 3,\ 4$. If $m_i(q,\gamma) < 0$ for all $i = 1,\ 2,\ 3,\ 4$, the E is locally asymptotically stable. If it exists any i for which $m_i(q,\gamma) > 0$, then, the equilibrium point E is unstable. So, the function $m_i(q,\gamma)$ for fractional-order systems has a similar effect as the real part of eigenvalues in integer system. Therefore, we can extend the Hopf bifurcation condition to the fractional systems by replacing $R_e(\lambda_i)$ with $m_i(q,\gamma) > 0$ as follows, compared with [11]:

- $m_{1,2}(q,\gamma^*) = 0$
- $\left. \frac{\partial m}{\partial \gamma} \right|_{\gamma=\gamma^*} \neq 0.$

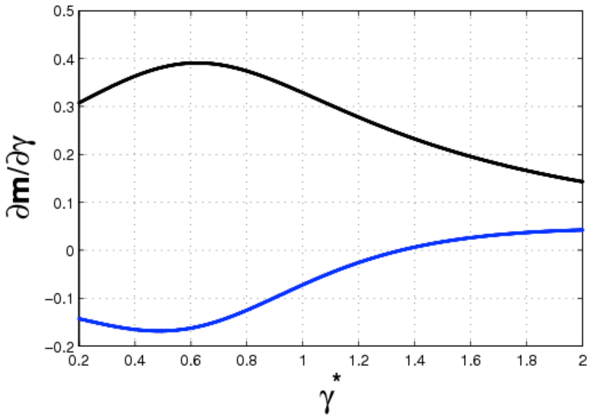
2.4 Hopf bifurcation versus the parameter β and the fractional order q

In this subsection, we consider the parameter values $(\sigma_1, \sigma_2, \sigma_3) = (1.25, 1, 1.75)$ for the search for the Hopf bifurcation around the equilibrium point O.

Fig.1a depicts the solution (q^*, γ^*) of equation $m(q,\beta) = 0$, while the black curve on Fig.1b recalls that $\left.\frac{\partial m_{1,2}}{\partial \gamma}\right|_{\gamma=\gamma^*} \neq 0$ for all $0 < \gamma^* < 2$, and for the blue curve on the same Fig.1b it can be noted that $\left.\frac{\partial m_{3,4}}{\partial \gamma}\right|_{\gamma=\gamma^*} \neq 0$ for all $0 < \gamma^* < 2$ except for $\gamma^* = 1.352$. We have $\left.\frac{\partial m}{\partial q}\right|_{q=q^*} = \frac{\pi}{2} \neq 0$, thus, the proposed fractional-order Hopf bifurcation conditions are verified for all pair (q^*, γ^*) solution of $m(q,\beta) = 0$, except for $(q^*, 1.352)$.



(a)



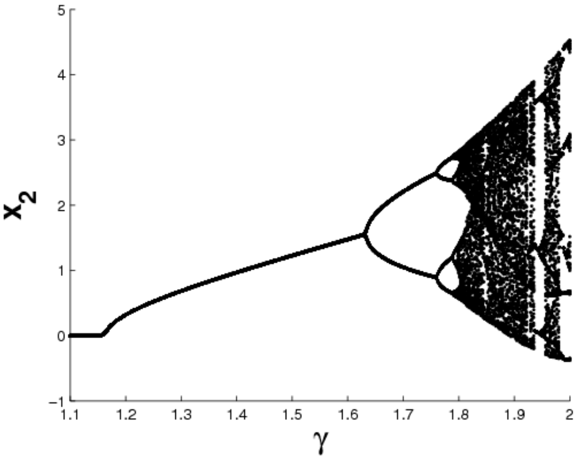
(b)

Fig. 1. Critical values γ^* versus the fractional order q^*, (a) this curve depicts the couples of values for which the Hopf bifurcation occurs in the system and evolution of $\left.\frac{\partial m_{1,2}}{\partial \gamma}\right|$ versus γ^* on (b).

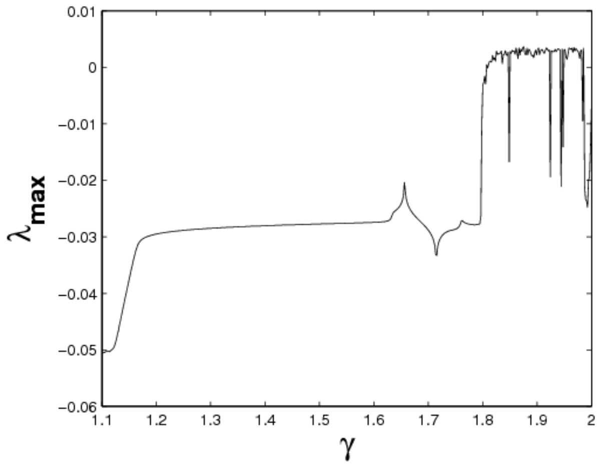
3. Numerical Results

3.1 Bifurcation and chaos versus the parameter γ

In this subsection, the dynamical behavior of the set of Eq.10 is numerically investigated by means of bifurcation diagram, and largest Lyapunov exponents, which measure the exponential rates of divergence or convergence of nearby trajectories in phase space. For γ taken as control parameter and the following other parameter values: fractional-order $q = 0.96$, $\sigma_1 = 1.25$, $\sigma_2 = 1.00$ and $\varepsilon = 1.175$, the critical Hopf bifurcation value is localized at $\gamma^* = 1.150$ (see Fig. 2.a), and confirmed by the diagram of the largest Lyapunov exponent presented in Fig. 2.b.



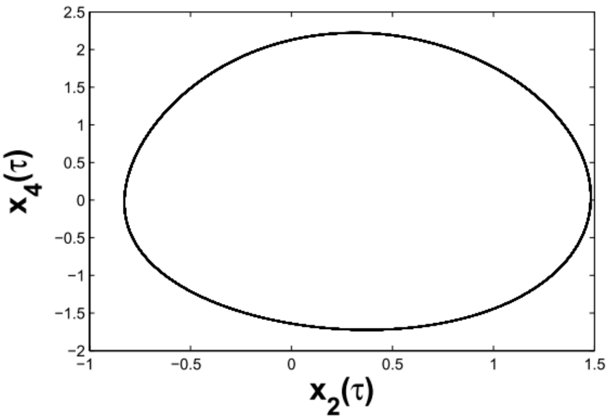
(a)



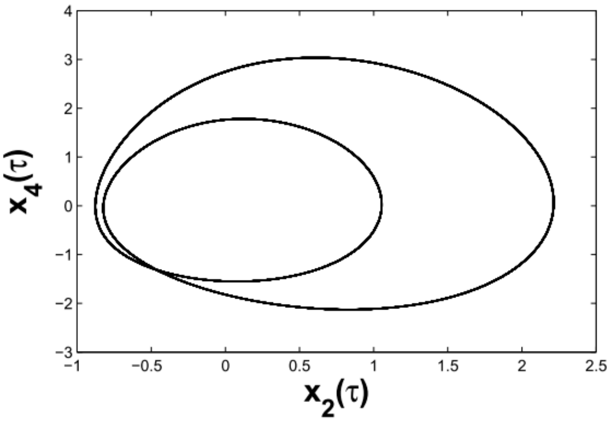
(b)

Fig. 2. (a) Bifurcation diagram expressing the dynamics of the system variable x_2, and (b) the largest Lyapunov exponent, both as a function of γ, with $q = 0.96$.

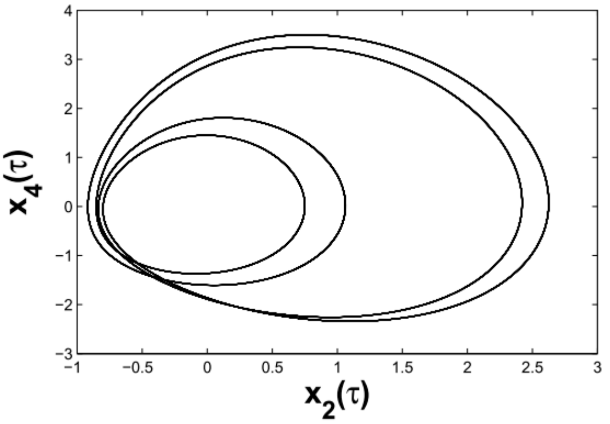
When $\gamma < 1.150$, the equilibrium point O is a locally asymptotically stable focus; the neighbors trajectories converge to O. This is supported by the negative sign of the largest Lyapunov exponents. For $1.150 < \gamma < 1.635$, Eq.10 undergoes a Hopf bifurcation as mentioned above. The fixed point O becomes unstable, and a period-one limit cycle appears. A period-two limit cycle follows for $\gamma \approx 1.635$, leading to a new bifurcation at $\gamma \approx 1.763$, as the system undergoes a period-four bifurcation. This bifurcations scenario continues through a period-height limit cycle for $\gamma \approx 1.791$ up to a critical value of $\gamma \approx 1.820$ corresponding to the appearance of a chaotic attractor. This chaotic behavior is confirmed by the existence of positive largest Lyapunov exponents. Figure 3 depicts the phase portraits presenting routes to chaos according to the above mentioned parameter values.



(a)



(b)



(c)
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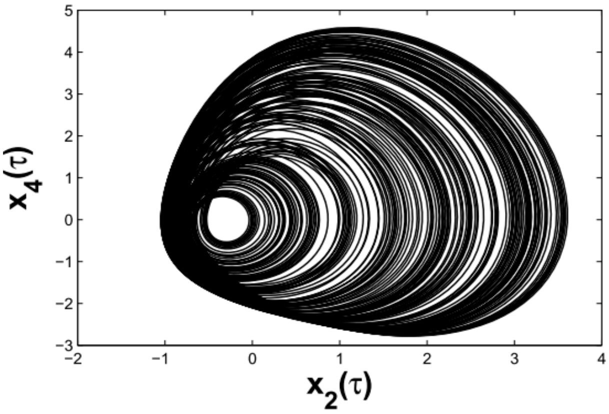
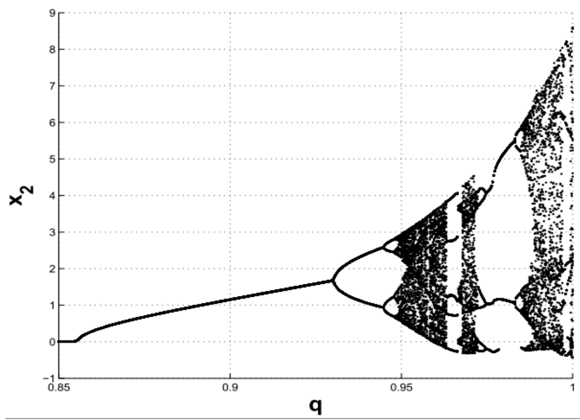


(d)

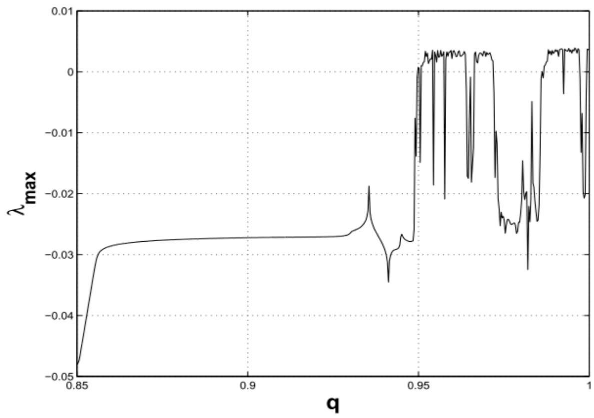
Fig. 3. Phase portrait of system (1) for different values of γ, with $q = 0.96$ (a) period-1 for $\gamma = 1.3$, (b) period-2 for $\gamma = 1.7$, (c) period-4 for $\gamma = 1.77$, (d) chaos for $\gamma = 1.9$.

3.2 Bifurcation and chaos versus the fractional order *q*

The fractional order q is taken as control parameter, while γ is fixed at γ = 1.9. The critical Hopf bifurcation value is localized at $q^* \approx 0.8557$, using the above proposed conditions. The resulting bifurcation diagram (Fig.4.a) for the second variable of the set of Eq.10 is plotted as a function of the fractional order q and the corresponding diagram, of the largest Lyapunov exponent is shown in Fig.4.b.



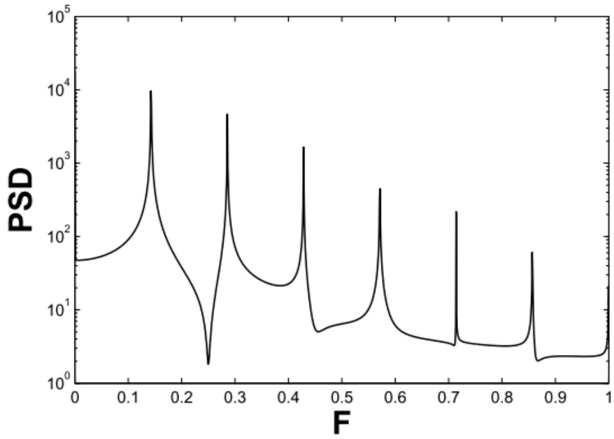
(a)



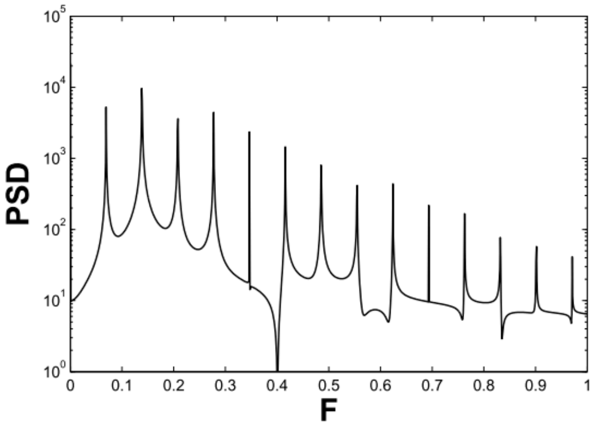
(b)

Fig. 4. (a) Bifurcation diagram expressing the dynamics of the system variable x_2, and (b) the largest Lyapunov exponent, both as a function of q, with $\gamma = 1.9$.

When $q < 0.8557$, the equilibrium point O is a locally asymptotically stable focus confirmed by the negative sign of the largest Lyapunov exponents; the neighbors trajectories converge to this origin. For $q = 0.8557$, the system called Eq.10 undergoes a Hopf bifurcation as mentioned above. The fixed points O becomes unstable, and a period-1 limit cycle appears for $0.8557 < q < 0.9307$. As the fractional order parameter nears the value $q \approx 0.9307$, a new bifurcation occurs for period-2 limit cycle. This is followed by a period-4 limit cycle at $q \approx 0.9454$. This bifurcation scenario continues up to a critical value $q \approx 0.953$ where a chaotic attractor appears, sustained by the existence of positive largest Lyapunov exponents. For a periodic steady state, all spikes in the power spectrum are harmonically related to the fundamental whereas a broadband noise like power spectrum is associated to a chaotic steady state. The periodicity of the attractor (i.e., total number of frequencies in a wave) is deduced by counting the number of spikes located at the left-hand side of the highest spike (the latter is included). Indeed, we have obtained the complete scenarios to chaos presented in Fig.5. Specifically, the following scenario was observed when monitoring the control parameter: fixed point behavior → period-1 → period-2 → period-4 → chaos.



(a)



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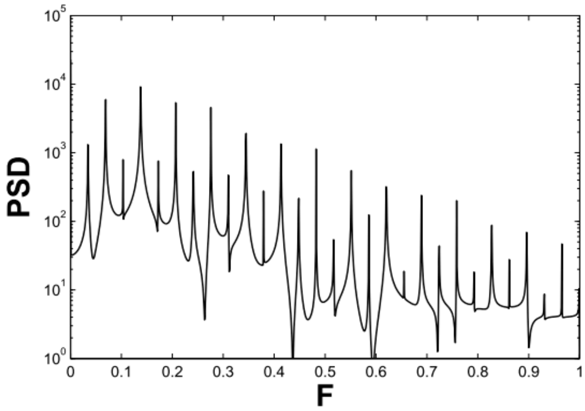

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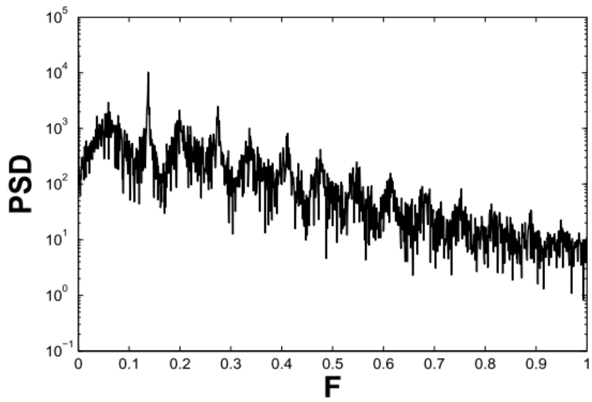

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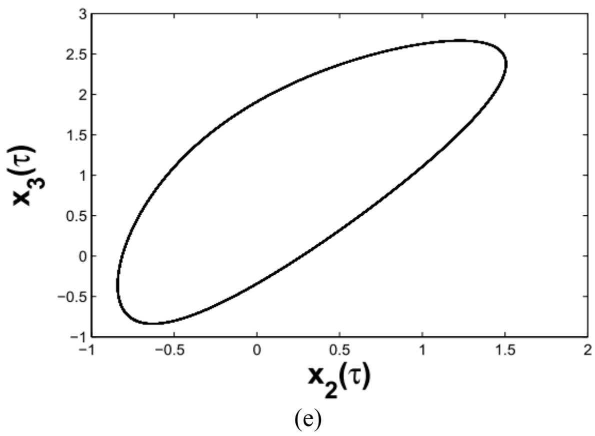

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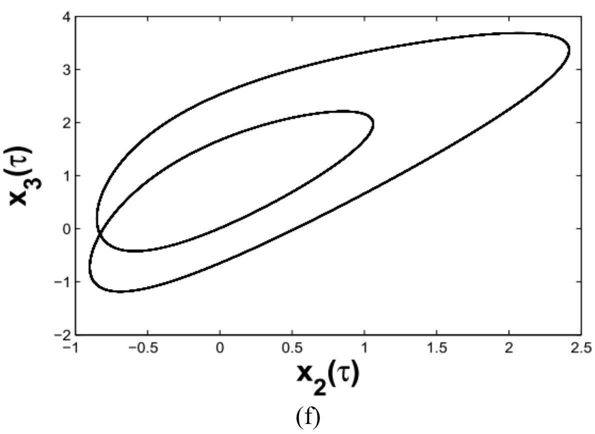

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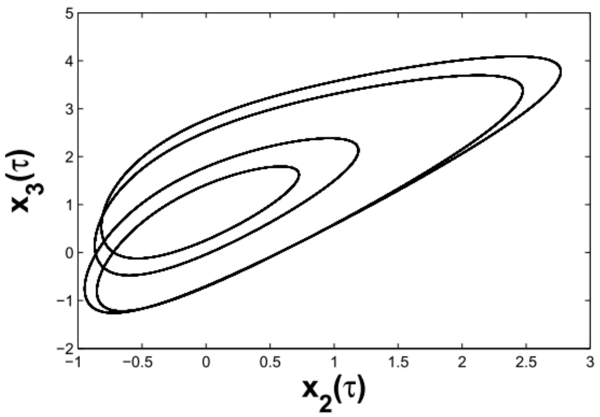

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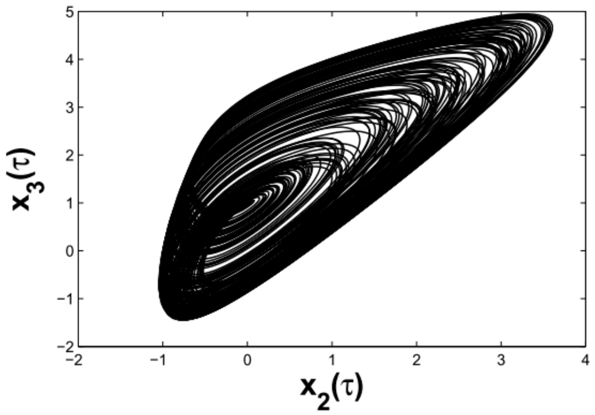

(h)

Fig. 5. corresponding power spectra of system (10) for different values of q, with γ = 1.9 and Phase portrait: (a) and (e) period-1 for q = 0.92, (b) and (f) period-2 for q = 0.94, (c) and (g) period-4 for q = 0.947, (d) and (h) chaos for q = 0.96.

4. Synchronization of Two Fractional-order Two-stage Colpitts Oscillators

4.1 Analytic results

This section is devoted to the synchronization of the drive and response commensurate fractional order of a two-stage Colpitts systems using nonlinear control, for q=0.96. The drive system is defined as follows:

$$\begin{cases} D^q x_1 = \sigma_1(x_4 - \gamma\phi(x_2 + x_3)) \\ D^q x_2 = x_4 \\ D^q x_3 = \sigma_2(x_4 - \gamma\phi(x_2)) \\ D^q x_4 = -x_1 - x_2 - x_3 - \varepsilon x_4 \end{cases} \tag{14}$$

Accordingly, the response system takes the following form:

$$\begin{cases} D^q y_1 = \sigma_1(y_4 - \gamma\phi(y_2 + y_3)) + u_1 \\ D^q y_2 = y_4 + u_2 \\ D^q y_3 = \sigma_2(y_4 - \gamma\phi(y_2)) + u_3 \\ D^q y_4 = -y_1 - y_2 - y_3 - \varepsilon y_4 + u_4 \end{cases} \tag{15}$$

with u_1, u_2, u_3 and u_4 the nonlinear controllers. By subtracting (14) from (15) and setting

$$\begin{cases} e_1 = y_1 - x_1 \\ e_2 = y_2 - x_2 \\ e_3 = y_3 - x_3 \\ e_4 = y_4 - x_4 \end{cases} \tag{16}$$

the following set of equations defining the errors is obtained:

$$\begin{cases} D^q e_1 = \sigma_1(e_4 + \gamma\phi(x_2 + x_3) - \gamma\phi(y_2 + y_3)) + u_1 \\ D^q e_2 = e_4 + u_2 \\ D^q e_3 = \sigma_2(e_4 + \gamma\phi(x_2) - \gamma\phi(y_2)) + u_3 \\ D^q e_4 = -e_1 - e_2 - e_3 - \varepsilon e_4 + u_4 \end{cases} \tag{17}$$

If we choose the control laws as described by the set of Eq.18 below,

$$\begin{cases} u_1 = \sigma_1\gamma\phi(y_2 + y_3) - k_1(y_1 - x_1) - \sigma_1\gamma\phi(x_2 + x_3) \\ u_2 = 0 \\ u_3 = \sigma_2\gamma\phi(y_2) - k_3(y_3 - x_3) - \sigma_2\gamma\phi(x_2) \\ u_4 = 0 \end{cases} \tag{18}$$

by substitution of (18) in (17) we obtain:

$$\begin{cases} D^q e_1 = \sigma_1 e_4 - k_1 e_1 \\ D^q e_2 = e_4 \\ D^q e_3 = \sigma_2 e_4 - k_3 e_3 \\ D^q e_4 = -e_1 - e_2 - e_3 - \varepsilon e_4 \end{cases} \tag{19}$$

Theorem 3

For $0 < q \le 1$, oscillators (14) and (15) will approach global synchronization for any initial condition with the control law defined by (18), if the conditions:

$$\begin{cases} k_1 > \dfrac{(1-\sigma_1)^2}{4\varepsilon} \quad \text{and} \\ \mathrm{k}_3 > 0 \end{cases} \tag{20}$$

are satisfied.

Proof

Construct a Lyapunov function:

$$V(t) = \frac{1}{2} e^T e \tag{21}$$

The time derivative of the Lyapunov function along the trajectories of system of Eq.19 is:

$$\frac{\partial^q V(t)}{\partial t^q} = e^T \frac{\partial^q e}{\partial t^q} = -\left[\left(k_1 - \frac{(1-\sigma_1)^2}{4\varepsilon}\right)e_1^2 + k_3 e_3^2 + \varepsilon\left(e_4 + \frac{(1-\sigma_1)}{2\varepsilon}e_1\right)^2\right].$$

It appears that the inequality $\dfrac{\partial^q V(t)}{\partial t^q} < 0$ is verified if

$$\begin{cases} k_1 > \dfrac{(1-\sigma_1)^2}{4\varepsilon} \quad \text{and} \\ k_3 > 0 \end{cases}$$

On the other hand, from Eq.19 we also have $\frac{\partial^q e}{\partial t^q} = Ae$.

Supposing that λ is one of the eigenvalues of matrix A, and that there should be a nonzero vector $\eta(\eta_1,\eta_2,....\eta_n)^T$ which is an eigenvector corresponding to the eigenvalues λ, then $A\eta = \lambda\eta$. The transposed matrix can be extracted from $\eta^T A\eta = \lambda\eta^T\eta$, $\eta^T A^T\eta = \bar{\lambda}\eta^T\eta$, therefore, $\eta^T\left(A + A^T\right)\eta = \left(\lambda + \bar{\lambda}\right)\eta^T\eta$.

Since $\frac{\partial^q V(t)}{\partial t^q} = e^T\left(A + A^T\right)e < 0$, $A + A^T$ is a negative definite matrix, and then $\left(\lambda + \bar{\lambda}\right) < 0$. The inequality $|\arg(\lambda)| > q\frac{\pi}{2}$ is obviously satisfied. According to the stability theory of fractional-order system [18, 19], the control law described as Eq.18 is stable, therefore, the fractional-order systems (14) and (15) can synchronize.

4.2 Numerical results

The system of errors defined by the set of Eq.19 is locally asymptotically stable if all the eigenvalues λ of the Jacobian matrix J below satisfy **theorem 2**.

$$J = \begin{bmatrix} -k_1 & 0 & 0 & \sigma_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_3 & \sigma_2 \\ -1 & -1 & -1 & -\varepsilon \end{bmatrix} \tag{22}$$

The eigenvalues equation is given as

$$\begin{aligned} p(\lambda) = {} & \lambda^{384} + (k_1 + k_3 + \varepsilon)\lambda^{288} + (1 + \sigma_1 + \sigma_2 + k_3k_1 + k_3\varepsilon + k_1\varepsilon)\lambda^{192} \\ & + (k_1 + k_3 + \sigma_1 k_3 + \sigma_2 k_1 + k_3k_1\varepsilon)\lambda^{96} + k_3k_1 \end{aligned} \tag{23}$$

Fig.6 defines the couples of points (k_1, k_3) obtained for $0 \le k_1 \le 3$ and $0 \le k_3 \le 5$ with a step of 0.1, for which **theorem 2** is verified and therefore the synchronization is achieved. This figure shows that for $k_1 = k_3 = 0$, the system is unstable. That is confirmed by the absence of dots on the two axis. **Theorem 2** shows that if $D = \frac{\pi}{2M} - \min_i\{|\arg(\lambda)|\} \ge 0$ the system is unstable else the system is globally asymptotically stable in the sense of Lyapunov [20]. Table1 gives a summary of the stability.

Table. 1. Stability domains of the system according to the value of parameters k_1, k_3, and D.

Value of the command parameter k_1	Value of the command parameter k_3	Value of $D = \frac{\pi}{2M} - \min_i\{\lvert\arg(\lambda)\rvert\} \ge 0$	Stability of the system (19)
$k_1 = 0$	$k_3 = 0$	$D = 0.0157 > 0$	unstable
$k_1 = 0$	$k_3 = 4$	$D = 0.0157 > 0$	unstable
$k_1 = 2$	$k_3 = 0$	$D = 0.0157 > 0$	unstable
$k_1 = 2$	$k_3 = 4$	$D = -0.0093 < 0$	stable

Thus, the synchronization between the drive (14) and response system (15) is stable for $k_1 > 0$ and $k_3 > 0$, (see Fig.6).

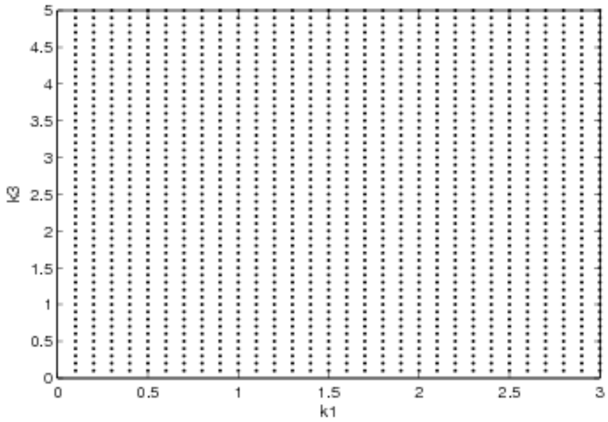

Fig. 6. Region of stability of the errors system for the whole couples of points k_1 and k_3 .

The drive (14) and response system (15) are numerically integrated with the parameter values σ_1 = 1.25, σ_2 = 1.00, γ = 1.9, ε = 1.175 and the control laws (19) using the feedback control gains $k_1 = 2$, and $k_3 = 4$. Fig.7 shows the behavior of synchronization errors between the drive (14), and response systems (15) with the controllers (18) for the fractional-order $q = 0.96$.

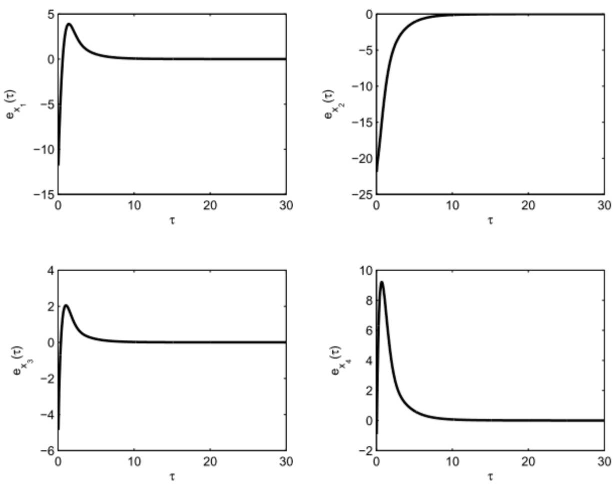

Fig. 7. Synchronization errors obtained for σ_1 = 1.25, σ_2 = 1.00, γ = 1.9, ε = 1.175, $q = 0.96$, $k_1 = 2$, and $k_3 = 4$. The system synchronizes for $\tau < 10$.

The synchronization state is depicted by Fig.8 where it can be seen that the synchronization on all the canals is achieved before the dimensionless time $\tau = 10$.

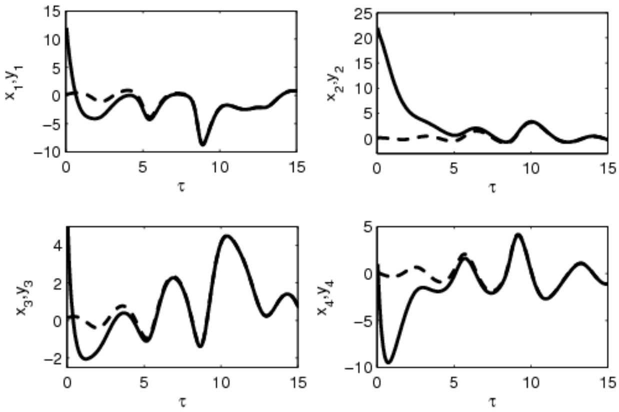

Fig. 8. Waveforms of state variables showing the synchronization process between the two coupled chaotic oscillators starting from different initial conditions (dot line: slave and solid line: master)

5. Conclusion

In this paper the dynamics and synchronization of a proposed four dimensional fractional-order two stage Colpitts oscillators have been investigated using analytical and numerical methods. The analytic method proved the existence of the Hopf bifurcation as well as the beach of the control parameter for which the system is stable. On the basis of fractional Lyapunov stability theory we determined with success the conditions under which the synchronization of two systems is achieved. For numerical simulation we used the Grünwald-Letnikov method, the largest Lyapunov exponents and the bifurcation diagrams to show the period-doubling bifurcation routes to chaos as well as the Hopf bifurcation. The numerical analysis validates the conditions of Hopf bifurcation. For the synchronization the numerical investigation validates also the analytic conditions which achieve synchronization. Numerical simulations have been used to show the effectiveness of the proposed synchronization techniques.

References

1. R. Hilfer, Applications of fractional calculus in physics, World Scientific, New Jersey (2001).
2. R.C. Koeller, J. Appl. Mech. **51**, 299 (1984).
3. H.H. Sun, A.A. Abdelwahab, and B. Onaral, IEEE Trans. Autom. Control. **29**, 441 (1984).
4. O. Heaviside, Electromagnetic Theory, Chelsea, New York (1971).
5. G.S. Mbouna Ngueuteu, and P. Woafo, Mechanics Research Communications **46**, 20 (2012).
6. I. Grigorenko, and E. Grigorenko, Phys. Rev. Lett. **91**, 034101 (2003).
7. K. Sun, J.C. Sprott, Elect Journal of Theoretical Physics. **6**, (22) 123 (2009).
8. L. Song, J.Y. Yang, S.Y. Xu, Nonlinear Analysis. **72**, 2326 (2010).
9. D. Chen, Y.Q. Chen, H. Sheng, Fractional variational optical flow model for motion estimation, in: Badajoz, The 4th IFAC Workshop Fractional Differentiation and Its Application, Spain, pp. 18–20 (Oct. 2010).
10. C. Julio, V. Gutiérrez, L.M. Carlos, J. Opt. A, Pure Appl. Opt. **10**, 1 (2008).
11. M.-S. Abdelouahab, N.-E. Hamri, J. Wang, Hopf bifurcation and chaos in fractional-order modified hybrid optical system, Nonlinear Dyn. (2011). doi 10.1007/s11071-011-0263-4.
12. O.A. Taiwo, O.S. Odetunde, AJESTR **1(2)**, 10 (2013).
13. Hadi Taghvafard , G.H. Erjaee, Commun Nonlinear Sci Numer Simulat **16(10)**, 4079 (2011).
14. Z.M. Odibat, Nonlinear Dyn. **60**, 479 (2010).
15. K. Zhang, H. Wang, H. Fang, Commun Nonlinear Sci Numer Simulat. **17**, 367 (2012).
16. J.G. Lu, Physica A. **359**, 107 (2006).
17. S.H. Hosseinnia, R. Ghaderi, A.N. Ranjbar, M. Mahmoudian, S. Momani, Computers and Mathematics with Applications. **59**, 1637 (2010).
18. E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, Journal of Mathematical Analysis and Applications. **325**, 542 (2007).
19. D. Matignon, Computational Engineering in Systems Applications, IEEE-SMC. **2**, 963 (1996).
20. A. Razminia, V.J. Majd, D. Baleanu, Advances in Difference Equations. **15**, 1 (2011).
21. M.S. Tavazoei, M. Haeri, M. Attari, S. Bolouki, M. Siami, J. Vib. Control. **15**, 803 (2009).
22. M.S. Tavazoei, M. Haeri, M. Attari, Automatica. **45**, 1886 (2009).
23. M.S. Tavazoei, Automatica. **46**, 945 (2010).
24. M. Mazandarani, A. Vahidian Kamyad, Commun Nonlinear Sci Numer Simulat **18**, 12 (2013).
25. L.O. Chua, and M. Itah, J. Circuits Syst. Comput. **3**, 93 (1993).
26. L.O. Chua, T. Yang, G.Q. Zhong, Int. J. Bifur. Chaos. **6**, 189 (1996).
27. G.R. Chen, X. Dong, From Chaos to Order, World Scientific, Singapore (1998).
28. L.M. Pecora, T.L. Carroll, Phys.Rev. Lett. **64**, 821 (1990).
29. J. Kengne, J.C. Chedjou, V.A. Fono, K. Kyamakya, On the analysis of bipolar transistor based chaotic circuits: case of a two-stage Colpitts oscillator, Nonlinear Dyn. (2011). doi 10.1007/s11071-011-0066-7.
30. I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999