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Research Article

Theoretical Solution of Elastic Foundation Beam based on the Principle of Minimum Complementary Energy

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Abstract

To simplify and solve the problem of the foundation beam, a half-space elastic foundation beam was selected, and the elastic foundation beam was systematically explored based on Boussinesq solution and the principle of minimum complementary energy. The step-by-step loading method was adopted to derive the settlement formula of the foundation beam. The superposition principle was used and the flexibility influence matrix was introduced to obtain the deformation energy of the foundation beam under the action of external force, and the complementary energy functional of foundation beam was established. Combined with the engineering example, the force and settlement of the foundation beam under the concentrated load and uniform load were calculated respectively, and the distribution characteristic of the ground reaction was analysed. Results show that the linear homogeneous differential equations with ground reaction as the basic unknown quantity are established and the method for solving the ground reaction is given. The comparison analysis proves that the new method is accurate and feasible. The new algorithm using the principle of minimum complementary energy can provide a reference for solving the similar engineering problems.

Keywords: Elastic foundation beam, Principle of minimum complementary energy, ground reaction, Settlement

1. Introduction

Foundation beams are basic structures in construction and are widely used in various civil and industrial buildings, such as highway construction, strip foundations in railway track design, grid beam foundations, and other building structures [1]. Since most engineering structures can be simplified into the calculation of elastic foundation beams, such as high-speed railway, underground pipelines, roof and floor of the roadway, and other engineering problems [2-4]. Although the calculation of elastic foundation beams is a classic research field, with the progress of the technology, new foundation models and calculation methods are emerging one after another [5-8].

At present, the classic theory of elastic foundation beam is widely used in construction fields, such as roads, railways and buildings [9-10]. As the scale of modern architectural design continues to increase, the weight of the building itself continues to increase, and the cost of the building foundation also continues to increase, which makes it becoming an important problem to design a foundation that meets the needs of the project within an economically reasonable range. To solve the ground reaction is equivalent to the contact problem. The difficulty in solving such problems is the calculation and solution of the interaction between the ground and the foundation beam. Therefore, the key is to choose a suitable foundation model and solution method. The studies on elastic foundation beam models mainly include Winkler foundation beam model [11], two-parameter foundation beam model [12], and half-space elastic foundation beam model [13].

The principle of minimum potential energy is based on displacement as the basic unknown quantity. The potential energy functional is established to solve the settlement, while the ground reaction is calculated from the displacement. However, it is rarely reported to solve the foundation beam based on the principle of minimum complementary energy. The reason is that the premise of using the stress variation method is to set the known stress not only to satisfy the stress boundary condition, but also to satisfy the equilibrium equation, which is difficult to achieve for general problems. But it is satisfactory for the half-space elastic foundation beam, and this method can directly solve the ground reaction, avoiding the error of the reaction obtained by the second calculation of the displacement.

Although an ideal foundation calculation model has been established, while with the development of large-scale construction projects, the interaction of ground and foundation beams, slabs and other foundations requires more in-depth theoretical studies. How to solve the internal force and displacement of the foundation beam more accurately and conveniently has become the key problem.

2. State of the art

Because the Winkler model requires few parameters, it has been widely used in the world. Many researchers have studied the Winkler foundation beam combined with the related engineering practice. Yu and Wang established a calculation model between the side walls of the tunnel based

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on the Winkler foundation beam model, they analyzed the distribution of deflection, bending moment and shear force of the side wall [14]. Ge and Xu established a numerical method to identify the void area between the foundation beam and the ground based on the Winkler foundation beam model [15]. According to the principle of the Winkler elastic foundation beam, the Winkler foundation model has some defects: The impact of shear deformation and the continuity of the ground are not considered, and the settlement of the foundation only occurs within the range of its base, which is obviously inconsistent with the reality. In addition, according to the Winkler's assumption, the coefficient K of the ground is a constant. But K has more influencing factors and it is not easy to determine [16-18].

For the elastic half-space foundation model, it considers the diffusion of stress and deformation and the interaction between adjacent loads, which is more in line with the engineering. Wang studied the analytical solutions of free beams on the elastic half-space ground under arbitrary lateral loads, including the beam deflection, bending moment and the contact reaction between the foundation beam and the ground [19]. Based on the Boussinesq's solution of a half-space elastic ground, Mi studied the local elastic solution of the displacement and stress on the surface loaded with nanomaterials under the load considering the effect of friction [20]. Kontomaris and Malamou solved the contact force between the rigid ball and the elastic body based on the elastic half-space model, and they analysed the deformation law and applicability of the theory under different radii of the ball [21]. Baraldi and Tullini proposed a simple and effective numerical model of bilateral frictionless contact between the flexible and rigid foundations in a three-dimension elastic half-space. They studied the laterally isotropic foundations parallel to the isotropic plane of the half-space boundary and obtained the settlement, the relationship between the inverse forces [22]. Tang et al. introduced the new parameters based on the half-space elastic model to give answers to satisfy various transverse anisotropic elastic models, and they specifically analysed the influencing factors of the transverse anisotropy [23-25]. The above mentioned researches all use the half-space elastic foundation model. But they all have the common shortcoming that the calculation model based on Boussinesq's solution is difficult to be solved directly by mathematical methods. Therefore, the difficulty in solving the half-space elastic foundation beam lies in the mathematical solution.

Among the methods for solving the elastic foundation beams, the principle of minimum potential energy based on the principle of energy variation has been widely used. Based on the principle of minimum potential energy and the analytical solution of the Winkler foundation beams, Li et al. constructed the Euler beam and Timoshenko beam elements [26-27]. Similarly, Guo et al. used the variational principle to solve the analytical solution of the bi-parameter elastic foundation beam considering the axial force [28]. Wang et al. analysed the deflection curve of half-space elastic foundation beams and compared them with other numerical methods to analyse the advantages and disadvantages of the approximate solution [29-31].

In this study, the elastic half-space foundation beam was taken as the research object, and the superposition principle was used to deduce the displacement reduction formula of the elastic half-space foundation under the load transfer action of the upper beam, and the complementary energy functional of the foundation beam was established. Based on the minimum complementary energy principle, the homogeneous linear equations with ground reaction as the basic unknown quantity were obtained. Finally, combining with a specific example, the solution was given and compared with other theoretical solutions to prove the correctness and applicability of the new method. The results can provide the theoretical basis and technical support for the similar practice.

The rest of this study is organized as follows. Section 3 describes the research methods. Section 4 gives the results and discussion, and finally, the conclusions are summarized in Section 5.

3. Methodology

The strain energy generated in the deformation process of the foundation under the action of the force was calculated, and then the expression of the complementary energy of the foundation was derived. Based on the deformation generated when the beam was bent, the expression of the complementary energy of the foundation beam was derived homogeneously. Finally, based on the energy principle, the complementary energy functional of the foundation beam was established.

3.1 Theoretical analysis of foundation beam

The foundation beam is analysed theoretically to study the applicability of the minimum complementary energy principle. As shown in Fig. 1, the equilibrium differential equation of the classical beam on elastic foundation is as Eq. (1).

$$E_b I \frac{d^4 w}{dx^4} = q(x) - p(x) \tag{1}$$

where, w is the deflection of the beam, E_b is the elastic modulus of the beam, I is the moment of inertia, $q(x)$ is the external load on the foundation beam, and $p(x)$ is the ground reaction.

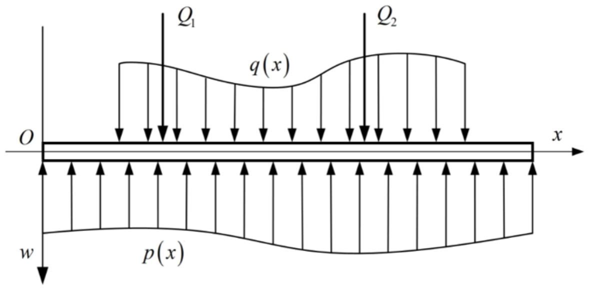


Fig. 1. Mechanical mode of the foundation beam.

Based on the Boussinesq's solution of the half-space elastic foundation model, the balance governing equation of the half-space elastic foundation beam is derived:

$$\frac{1-\mu_f^2}{\pi E_f} \cdot \frac{d^4}{dx^4}\left[\iint_{\Omega} \frac{p(\xi,\varsigma)d\xi d\varsigma}{\sqrt{(x-\xi)^2(x-\varsigma)^2}}\right] = \frac{q(x)-p(x)}{E_b I} \tag{2}$$

where, E_f is the elastic modulus of ground, μ_f is the Poisson's ratio of ground. In Eq. (2), $p(x)$ is the unknown function which satisfies the equilibrium condition:

$$\int p(x)dx = \int q(x)dx \tag{3}$$

Since it is difficult to solve the Eq. (2) mathematically in combination with the static boundary conditions. For the foundation beam, the external load on the foundation beam and the ground reaction are shown in Fig. 1, which satisfies the force balance condition, that is, satisfies the known force boundary condition. The deduction of the complementary energy from the balance equation of the foundation beam is based on the equivalent relationship between the deflection equation and the bending moment equation, which shows that each cross-section in the foundation beam satisfies the balance differential equation. For the ground, based on the Boussinesq's solution to solve the deformation of the half-space elastic ground under the action of force, which shows that any point in the ground satisfies the balance differential equation and the boundary conditions of force and displacement. The settlement and force at infinity of the ground tend to be zero when the weight of the ground is not taken into account, the principle of minimum complementary energy is suitable for the solution of the foundation beam.

3.2 Settlement analysis of half-space elastic ground

To establish the complementary energy functional of the foundation beam based on the Boussinesq's solution, the settlement formula of the foundation beam under the transferred load was deduced, and the complementary energy formula of the half-space elastic ground under the load was further obtained.

As shown in Fig. 2, the contact part of the beam and ground is divided into n numbers of equal size elements along the length direction by using the method similar to the finite element. Let each cell be 2a in length and 2b in width. For the beam with rectangular section, the change of the load on the groud in the beam width is ignored. Referring to the groud reaction shown in Fig. 1, the distributed force on the foundation is recorded as $p(x)$. When the element size is small enough, the uniform force on each element can be regarded as a set of concentrated forces. As seen from Fig. 2, the uniform force of $4abp$ can be equivalent to the concentrated force of $\overline{p}_1$. So the complementary energy functional of the foundation beam is deduced, the formula for the settlement of the loaded element is deduced.

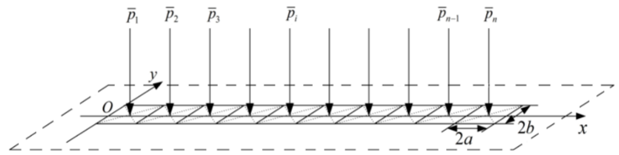
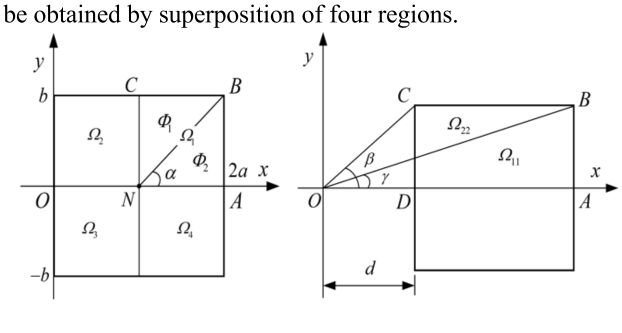


Fig. 2. Schematic diagram of the equivalent concentrated forces on the foundation beam.

As shown in Fig. 3(a), taking any unit in Fig. 2 as the research object and set N as the center of the element. The whole rectangular region is divided into four equal sized regions. The $ABCN$ is composed by two parts: Φ_1 and Φ_2. The displacement at point N under the action of load in $ABCN$ region is calculated, and then the displacement expression at the center point of the rectangular element can be obtained by superposition of four regions.



(a) Settlement of element center (b) Loading area outside element
Fig. 3. Settlement calculation diagram of the element.

The deformation region of the element is represented by Ω. Taking a micro element in the Ω area, the load on the micro element is $p$dxdy and the load on the micro element can be regarded as the concentrated load. Based on the settlement formula under the concentrated load on the half-space elastic ground model, the vertical displacement at the center point N of the element under the concentrated load can be calculated as Eq. (4).

$$w=\frac{p\left(1-\mu_0^2\right)}{\pi E_0}\iint_{\Omega}\frac{dxdy}{\sqrt{(x-a)^2+y^2}} \tag{4}$$

where, E_0 and μ_0 is the elastic modulus and the Poisson's ratio of the ground, respectively.

Setting N point as the point in the polar coordinate system, and X-axis is the polar axis of the polar coordinate system, the combination of Φ_1 and Φ_2 into Ω_1 is regarded as a region. The rectangular cell $ABCN$ is shown in Fig. 3(a), ∠BNA=α, NA=CB=a, BA=CN=b, so the interval range of Φ_1 and Φ_2 is as following:

$$\begin{aligned}&\Phi_1: 0\le\theta\le\alpha,\ \ 0\le\rho\le a\sec\theta\\&\Phi_2: \alpha\le\theta\le\frac{\pi}{2},\ \ 0\le\rho\le b\csc\theta\end{aligned} \tag{5}$$

The whole rectangular element is divided into four parts: Ω_1, Ω_2, Ω_3, and Ω_4. Since $\Omega_1=\Phi_1+\Phi_2$. So Eq. (4) can be calculated as follow:

$$\iint_{\Omega_1}\frac{dxdy}{\sqrt{(x-a)^2+y^2}}=\iint_{\Phi_1}\frac{dxdy}{\sqrt{(x-a)^2+y^2}}+\iint_{\Phi_2}\frac{dxdy}{\sqrt{(x-a)^2+y^2}} \tag{6}$$

Based on the polar coordinate, Eq. (6) can be solved as Eq. (7).

$$\begin{aligned}&\iint_{\Omega_1}\frac{dxdy}{\sqrt{(x-a)^2+y^2}}\\&=\int_0^{\alpha}d\theta\int_0^{a\sec\theta}\frac{1}{\rho}\cdot\rho d\rho+\int_{\alpha}^{\frac{\pi}{2}}d\theta\int_0^{b\csc\theta}\frac{1}{\rho}\cdot\rho d\rho\\&=a\ln\left|\sqrt{1+\tan^2\alpha}+\tan\alpha\right|-b\ln\left|\sqrt{1+\cot^2\alpha}-\cot\alpha\right|\end{aligned} \tag{7}$$

Setting cotα=a/b=m, then Eq. (7) can be simplified as following:

$$\iint_{\Omega_1}\frac{dxdy}{\sqrt{(x-a)^2+y^2}}=b\left[m\ln\frac{1+\sqrt{1+m^2}}{m}+\ln\left(m+\sqrt{1+m^2}\right)\right] \tag{8}$$

Substituting Eq. (8) into Eq.(4), then Eq. (9) is:

$$w=\frac{bp\left(1-\mu_0^2\right)}{\pi E_0}\left[m\ln\frac{1+\sqrt{1+m^2}}{m}+\ln\left(m+\sqrt{1+m^2}\right)\right] \tag{9}$$

According to the theory of elasticity, the settlement at the center point N of the element is the total settlement of the four parts at N, then Eq. (10) is:

$$w_N = \frac{4bp(1-\mu_0^2)}{\pi E_0}\left[m\ln\frac{1+\sqrt{1+m^2}}{m} + \ln\left(m+\sqrt{1+m^2}\right)\right] \tag{10}$$

The calculation of the settlement caused by the load outside the element area is shown in Fig. 3(b). Similarly, taking a micro-element in *ABCD*, and regarding the force on the micro-element as a concentrated force too. So the settlement of the force on the element *ABCD* to the point *O* is the superposition of two parts. The settlement at *O* can also be expressed by $\overline{w}$ as Eq. (11).

$$\overline{w} = \frac{p(1-\mu_0^2)}{\pi E_0}\iint_\Omega \frac{dxdy}{\sqrt{(2a-x+d)^2+y^2}} \tag{11}$$

where, d is the distance between the center point *O* of the element for settlement calculation and the nearest boundary *D* of the loaded element.

The specific derivation process is omitted and the solution in the polar coordinate is calculated as Eq. (12).

$$\overline{w} = \frac{2p(1-\mu_0^2)}{\pi E_0}\begin{pmatrix} 2a\psi_2(d+2a) \\ +b[\psi_3(d)-\psi_3(d+2a)] \\ +d[\psi_2(d+2a)-\psi_2(d)] \end{pmatrix} \tag{12}$$

where, $\psi_2(t)$ and $\psi_3(t)$ in Eq. (12) are expressed as following:

$$\psi_2(t) = \ln\frac{b+\sqrt{t^2+b^2}}{t},\ \ \psi_3(t) = \ln\sqrt{t^2+b^2} - t$$

3.3 Complementary energy analysis of ground settlement

When the foundation beam is under the loading, it is assumed that the ground and the foundation beam are in complete contact during the deformation process, and only the settlement in the vertical direction of the ground is considered regardless of the interlayer friction. According to the equivalent principle in elasticity, the deformation energy is independent of the order of the force applied by the elastic body, and depends on the external force and the final deformation of the elastic body. For the deformation energy of the ground, it is assumed that the concentrated loads are loaded on the ground in order from left to right. Assuming the work done by the concentrated load $\overline{p}_i$ on the ground is $\overline{U}_i(i=1,2,\cdots,n)$ as Eq. (13).

$$\begin{aligned}
\overline{U}_1 &= \frac{1}{2}w_{11}\overline{p}_1 + (w_{12}\overline{p}_1 + \cdots + w_{1n}\overline{p}_1) = \frac{1}{2}w_{11}\overline{p}_1 + \sum_{i=2}^{n} w_{1n}\overline{p}_1 \\
\overline{U}_2 &= \frac{1}{2}w_{22}\overline{p}_2 + (w_{22}\overline{p}_2 + \cdots + w_{2n}\overline{p}_2) = \frac{1}{2}w_{22}\overline{p}_2 + \sum_{i=3}^{n} w_{2n}\overline{p}_2 \\
&\cdots\cdots \\
\overline{U}_i &= \frac{1}{2}w_{ii}\overline{p}_i + \sum_{j=i+1}^{n} w_{ij}\overline{p}_i
\end{aligned} \tag{13}$$

where, w_{ij} is the settlement at the center of the element i caused by the force on the element j, as shown in Fig. 4.

According to the superposition principle of elasticity, the sum of the work done by each concentrated load is the work of all concentrated load. Because the relationship of the force-displacement is linear, the complementary energy and the strain energy are equal. The complementary energy of the ground is the work done by all concentrated loads, and which can be expressed by U_1 as Eq. (14).

$$U_1 = \sum_{i=1}^{n}\overline{U}_i = \frac{1}{2}\sum_{i=1}^{n} w_{ii}\overline{p}_i + \sum_{i=1}^{n}\sum_{j=i+1}^{n} w_{ij}\overline{p}_i \tag{14}$$

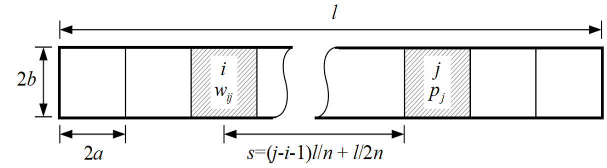


Fig. 4. Schematic diagram of w_{ij}.

For each element, there is a linear relationship between the concentrated load $\overline{p}_i$ and w_{ii}. If the size of each cell is the same, the scale factor k_{ii} is equal as Eq. (15).

$$\overline{p}_i = k_{ii}w_{ii} \quad (i,j=1,2,\cdots,n) \tag{15}$$

Since there is a similar linear relationship between the w_{ij} and $\overline{p}_j$, the relationship coefficient is recorded as k_{ij}:

$$\overline{p}_j = k_{ij}w_{ij} \quad (i,j=1,2,\cdots,n) \tag{16}$$

Setting $c_{ij}=1/k_{ij}$, and c_{ij} is the flexibility influence coefficient of j element to i element, combined with Eqs. (15) and (16), the complementary energy of the ground can be calculated as Eq. (17).

$$\begin{aligned}
U_1 &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\overline{p}_i\overline{p}_j \\
&= \frac{1}{2}\sum_{j=1}^{n} c_{ii}\overline{p}_i^{\,2} + \sum_{i=1}^{n}\sum_{j=i+1}^{n} c_{ij}\overline{p}_i\overline{p}_j
\end{aligned} \tag{17}$$

The vertical settlement w_{ii} of the element can be calculated based on the Boussinesq's solution in polar coordinate as Eq. (18).

$$w_{ii} = \frac{\overline{p}_i(1-\mu_0^2)}{\pi aE_0}\left[m\ln\frac{1+\sqrt{1+m^2}}{m} + \ln\left(m+\sqrt{1+m^2}\right)\right] \tag{18}$$

As shown in Fig. 4, the settlement w_{ij} caused by the equivalent concentrated force $\overline{p}_j$ at the center point x_i of other elements can be calculated as Eq. (19).

$$w_{ij} = \frac{\overline{p}_j(1-\mu_f^2)}{2\pi abE_f}\left[2a\ln\frac{b+o}{s+2a} + b\ln\frac{\lambda-s}{o-s-2a} + s\ln\frac{s(b+o)}{(s+2a)(b+\lambda)}\right] \tag{19}$$

where, s is the distance between the center of the i element and the left boundary of j element. $o=\sqrt{b^2+(s+2a)^2}$; $\lambda=\sqrt{b^2+s^2}$; $s=(j-i-1)\dfrac{l}{n}+\dfrac{l}{2n}$.

By introducing the flexibility influence matrix, Eq. (13) is written in the matrix form as Eq. (20).

$$U_1 = \frac{1}{2}\begin{pmatrix}\overline{p}_1\\ \overline{p}_2\\ \vdots\\ \overline{p}_n\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} c_{11} & c_{21} & \cdots & c_{n1}\\ c_{12} & c_{22} & \cdots & c_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ c_{1n} & c_{2n} & \cdots & c_{nn}\end{pmatrix}\begin{pmatrix}\overline{p}_1\\ \overline{p}_2\\ \vdots\\ \overline{p}_n\end{pmatrix} \tag{20}$$

The ratio of settlement at the center point of element i to that at the center point of element j is denoted by q_{ij}, which has the following characteristics:

$$q_{ii} = \frac{w_{ii}}{w_{ii}} = 1 \quad (i = 1,2,\cdots,n) \tag{21}$$

$$q_{ii} = \frac{w_{ij}}{w_{jj}} \quad (i = 1,2,\cdots,n; j = 1,2,\cdots,n; j > i) \tag{22}$$

where, q_{ij} is related to the relative position of the element and not related to the load size.

According to the half-space elastic ground model, q_{ij} is symmetry: $q_{ij}=q_{ji}$. From Eqs. (18), (19), and (20), q_{ij} can be obtained.

According to the relative position of the element, and after calculation and verification, the general equation of the influence coefficient q_{ij} can be calculated as Eq. (23).

$$q_{ij} = q[i,j] = q[n, n+j-i] \tag{23}$$

By introducing the influence coefficient of flexibility and combining Eqs. (6) and (12), w_{ij} can be rewritten as Eq. (24).

$$w_{ij} = c_{ij}\bar{p}_j = q_{ij}w_{jj}; \quad c_{ij} = \frac{q_{ij}}{k_{jj}} \tag{24}$$

Eq. (20) is rewritten as Eq. (25).

$$U_1 = \frac{1}{2k}\begin{pmatrix}\bar{p}_1\\ \bar{p}_2\\ \vdots\\ \bar{p}_n\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n}\\ q_{21} & q_{22} & \cdots & q_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ q_{n1} & q_{n2} & \cdots & q_{nn}\end{pmatrix}\begin{pmatrix}\bar{p}_1\\ \bar{p}_2\\ \vdots\\ \bar{p}_n\end{pmatrix} = \frac{1}{2k}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{K}_f\boldsymbol{P} \tag{25}$$

Since the scale factor k_{ii} of the equal size element is equal, it is expressed by k in Eq. (25).

The complementary energy expression of the ground is obtained as Eq. (26).

$$U_1 = \frac{1}{2k}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{K}_f\boldsymbol{P} \tag{26}$$

3.4 Complementary energy analysis of foundation beam

Assuming that the deflection curve of the foundation beam and the settlement of the ground are equal everywhere, ignoring the friction between the foundation beam and the ground. It is known that the length of the foundation beam is l, the uniformly distributed load $q(x)$ and the concentrated load Q_i (i=1, 2, ⋯, n) act on the foundation beam, the deflection curve of the foundation beam is $w(x)$, and the complementary energy (the deformation energy of the beam) of the foundation beam is U_2. It can be calculated as Eq. (27).

$$U_2 = \frac{1}{2}E_bI\int_0^l[w''(x)]^2dx \tag{27}$$

According to the relationship between the second derivative of deflection equation and moment equation as Eq. (28):

$$w''(x) = \frac{M(x)}{E_bI} \tag{28}$$

So Eq. (27) is rewritten as Eq. (29).

$$U_2 = \frac{1}{2E_bI}\int_0^l[M(x)]^2dx \tag{29}$$

where, $M(x)$ can be expressed according to the force of the foundation beam as shown in Fig. 1. The bending-moment equation $M(x)$ can be calculated as Eq. (30).

$$M(x) = \int_0^x(x-\varsigma)[p(x)-q(x)]d\varsigma \tag{30}$$

The total complementary energy of the foundation beam system includes the deformation energy of the ground and the foundation beam. The complementary energy functional V can be calculated as Eq. (31).

$$V = U_1 + U_2 = \frac{1}{2k}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{K}_f\boldsymbol{P} + \frac{1}{2E_bI}\int_0^l[M(x)]^2dx \tag{31}$$

As shown in Eq. (31), the complementary energy functional is a function of $\bar{p}_i$. Based on the principle of minimum complementary energy, the ground reaction of the system can be calculated as Eq. (32).

$$\frac{\partial V}{\partial \bar{p}_i} = 0 \quad (i = 1,2,\cdots,n) \tag{32}$$

According to the precondition of half-space elastic foundation beam, the known deformation compatibility condition should be satisfied, and the deflection of each point on the beam is equal to the settlement of the corresponding part of the ground. In addition, the ground reaction must satisfy the equilibrium condition as Eq. (33).

$$\int_0^l Q(x)dx = \sum \bar{p}_i \quad (i = 1,2,\cdots n) \tag{33}$$

4. Results and discussion

As shown in Fig. 5, supposing a foundation beam is in complete smooth contact with the ground, the elastic modulus and the bending stiffness of the foundation beam is E_b = 2 GN/m^2 and E_bI_b = 145.8 MN·m^2 respectively, the length and width of which are l = 6 m and $2b$ = 0.7 m respectively. The elastic modulus and Poisson's ratio of the ground is E_f= 6.5 MPa and μ_f= 0.25, respectively. Assuming working condition 1: The central part of the foundation beam is subjected to a concentrated force Q = 80 kN; Working condition 2: There is uniform load $q(x)$=20 kN/m on the foundation beam.

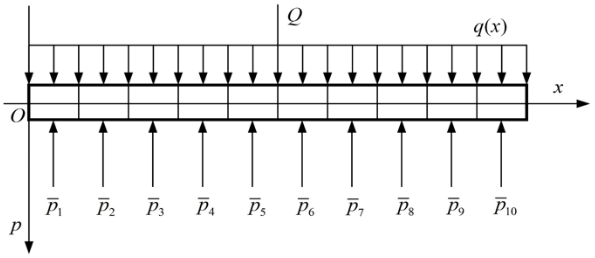


Fig. 5. The forces of the foundation beam.

According to the above mentioned method, the contact part between the foundation beam and ground is divided into 10 equal large elements along the length in Fig. 5. Assuming the concentrated force on each element is is $\bar{p}_i$ $(i=1,2,\cdots n)$, the equation is established by Eq. (32), and the equivalent concentrated load on each element is obtained. The solution process is to write a self-development calculation program for calculation. Tables 1 and 2 show the solution results of the foundation beam under the concentrated and uniform loads, respectively.

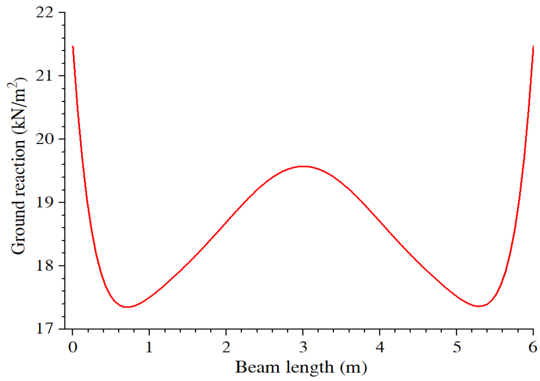
Table 1. The concentrated load on the element.

Equivalent concentrated force	$\bar{p}_1$	$\bar{p}_2$	$\bar{p}_3$	$\bar{p}_4$	$\bar{p}_5$
Result (kN)	9.02	7.32	7.58	7.91	8.18
Equivalent concentrated force	$\bar{p}_6$	$\bar{p}_7$	$\bar{p}_8$	$\bar{p}_9$	$\bar{p}_{10}$
Result (kN)	8.18	7.91	7.58	7.32	9.02

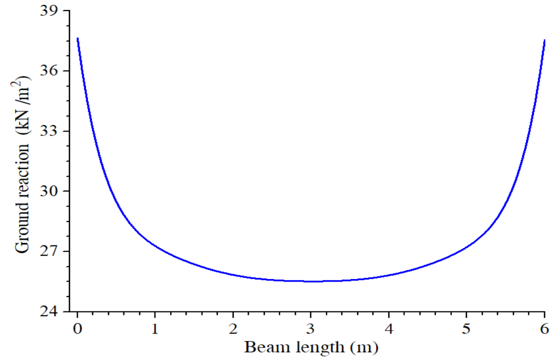
Table 2. The uniform load on the element.

Equivalent concentrated force	$\bar{p}_1$	$\bar{p}_2$	$\bar{p}_3$	$\bar{p}_4$	$\bar{p}_5$
Result (kN)	15.81	11.57	11.07	10.82	10.73
Equivalent concentrated force	$\bar{p}_6$	$\bar{p}_7$	$\bar{p}_8$	$\bar{p}_9$	$\bar{p}_{10}$
Result (kN)	10.73	10.82	11.07	11.57	15.81

The concentrated force in Table 1 is equivalent to the average force on each element as follow: $\bar{p}_i$=4*abp*. Then, the distribution curve of the ground reaction is obtained by fitting with the least square method. As shown in Fig. 6, there is a large ground reaction at the end of the foundation beam. In the deformation process of the half-space elastic ground and foundation beam, when the bending stiffness of the foundation beam is relatively large, although the soil around the end of the beam is not subject to the load, it has the effect of supporting constraint on the end of the beam, and the end of the beam is strongly supported, resulting in the stress concentration phenomenon of sharp increase of the contact force.



(a) The concentrated load

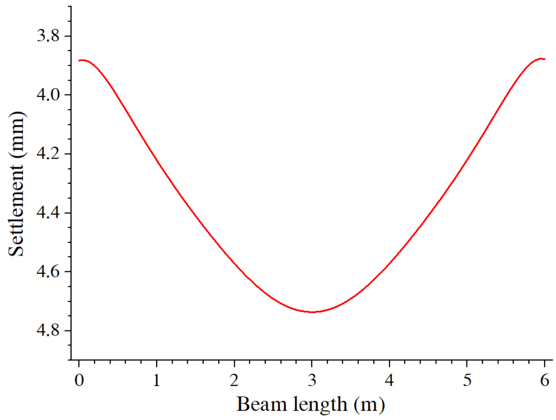


(b) The uniform load

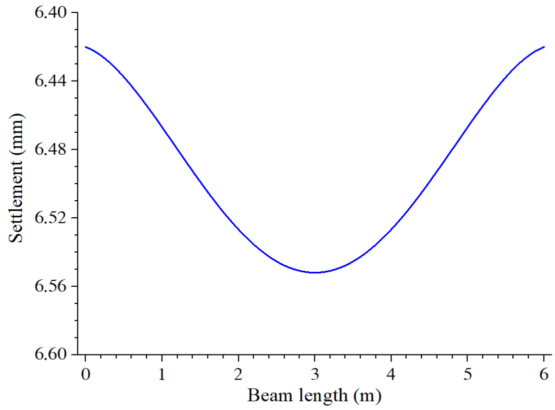
Fig. 6. The fitting curve of the ground reaction.

Based on the results of the obtained ground reaction, combined with Eqs. (18) and (19) to solve the settlement of the elastic foundation beam under the load, it is conducive to the least square method to fit the curve. The results are shown in Fig. 7.

In addition, based on the comparative analysis of the calculation examples [30], the method proposed in literature [30] is based on the generalized variational principle to solve the elastic foundation beam. The main calculation parameters are as follow: the elastic modulus of the beam is E_b= 2.1 GN/m^2. The length, the width and the height of the foundation beam is l = 6 m, $2b$ = 0.7 m, and h = 0.4 m, respectively. The elastic modulus and Poisson's ratio of the ground is E_f= 1.5 MPa and μ_f=0.3, respectively. The solution is carried out under the action of the concentrated force Q = 45 kN in the middle of the foundation beam and uniformly distributed load $q(x)$=15 kN/m, respectively.



(a) The concentrated load

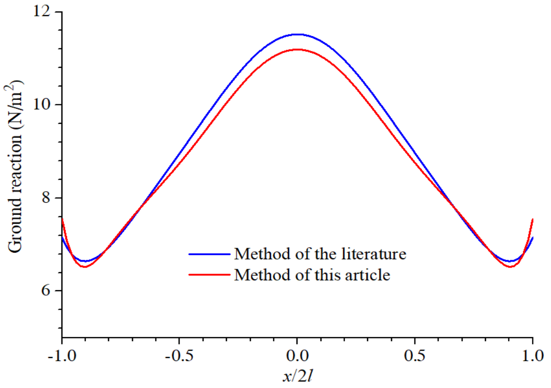


(b) The uniform load

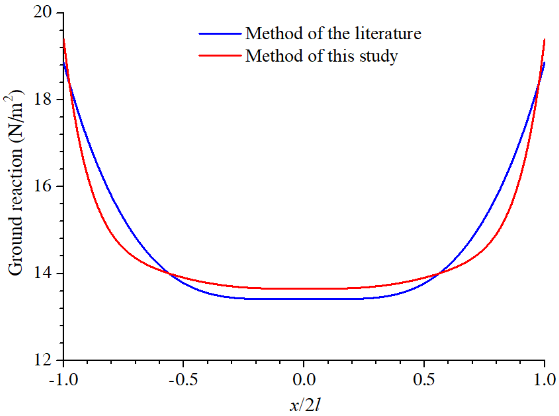
Fig. 7. The fitting curve of the settlement.

As shown in Fig. 8, the results of the two methods are very similar, which verifies accuracy of the new method.

However, after the detailed comparative analysis, the complicated formula derivation is carried out in the literature [30]. The method in this study is more intuitive to solve the ground reaction directly. In addition, the new solution is simpler, and the derivation process of the formula is simplified. The calculation is greatly reduced and it can be solved by the programming of the mathematical software.



(a) The concentrated load



(b) The uniform load

Fig. 8. Comparative analysis of the ground reaction.

5. Conclusions

To simplify and solve the problem of the foundation beam, based on the principle of minimum complementary energy, the interaction of the half-space elastic foundation beam system under the concentrated and uniform load was analysed. The main conclusions are as following:

(1) The polar coordinate solution of the settlement of the half-space elastic ground is derived. The expressions of the deformation energy of the half-space elastic ground and foundation beam are derived by step-by-step loading. The complementary energy functional of the half-space elastic foundation beam system is established.

(2) Not only the homogeneous equations are established, but also the analytical solution and settlement of the foundation beam system under the uniform and concentrated loads are solved respectively. The stress concentration at the end of the foundation beam under the action of load is explained reasonably.

(3) The feasibility and accuracy of the new method are verified. Meanwhile, the disadvantages of other methods and the complicated calculation process are avoided. The new method can be used in the calculation of the similar engineering, which calculation process is simple and the accuracy is high.

Although the free foundation beam at both ends is studied, but for some practical problems, there are constraints at the end of the beam, so the simply supported foundation beam or fixed foundation beam must also be further studied. The settlement formula of any point in the element should be more accurate deduced.

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