Journal of Engineering Science and Technology Review 11 (2) (2018) 121 - 125

Research Article

JOURNAL OF
Engineering Science and Technology Review

www.jestr.org

WKB Results for the Rotating Thick-Walled Incompressible Elastic Tubes

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Received 13 December 2017; Accepted 17 February 2018

Abstract

In this paper, we investigated deformation in rotating thick-walled circular cylinders of incompressible, isotropic, neo-Hookean material. We apply the WKB method to the bifurcation analysis of cylinder shell which is subjected an external pressure. In the all mode numbers find that present compression inner layer. Amongst other results, we obtained present compression outer layer. Finally, we compared the asymptotic results with different mode numbers.

Keywords: WKB analysis; Wave speed; Incompressible

1. Introduction

The bifurcation with respect to a circular cylindrical configuration of a circular cylinder of incompressible rotating elastic material has been examined by several previous studies and includes a wide theory of incremental nonlinear elasticity. In 1980, Haughton and Ogden [1] derived an equation relating angular speed, axial loading, and elastic properties and examined its implications given known material behavior. Haughton [2] proved that pure tensional, longitudinal and breathing mode vibrations cease to exist when rotation is initiated. Ogden [3] studied critical values of the tension at which bifurcation occurred for a general form of the strain-energy function. In 2001, Bigoni and Gei [4] investigated bifurcations in velocities from a state of homogeneous axisymmetric deformation for a coated elastic cylinder that was subjected to axial tension or compression. Dorfmann et al. [5] developed a concise general theory of nonlinear magnetoelasticity to analyze the mechanical response of a circular cylindrical tube under steady rotation with respect to its axis in an azimuthal magnetic field and a solid circular cylinder also under steady rotation about its axis in an axial magnetic field. The *WKB* method is widely applied to fluid stability problems, and [6] provide an excellent review of its application in solving the eigenvalue problem involving the bifurcation analysis of a spherical shell of arbitrary thickness.

Fu and Sanjaranipour [7] applied the fore-mentioned method to the stability analysis of an everted cylindrical tube. Sanjaranipour [8,9] extended the study by Fu and applied the method to the buckling analysis of Varga and a neo-Hookean material cylindrical shell of arbitrary thickness subjected to external hydrostatic pressure.

Haughton and Chen [10] applied the method for the bifurcation analysis of everted cylindrical and spherical shells. In 2007, Coman and Bassom [11] applied the *WKB* and boundary layer asymptotic methods to examine these issues and compared the wrinkling of a pre-stressed annular thin film in tension.

Coman and Destrade [12] used the bifurcation of an incompressible neo-Hookean thick hyperelastic plate and reduced a fourth-order linear eigenproblem that displayed multiple turning points. Recently Sanjaranipour et al. [13] used the *WKB* method with repeated roots to describe the angle of bending in addition to the azimuthal shear.

Section 2 describes the analysis of circular cylindrical configurations that is performed with respect to a completely general form of incompressible isotropic elastic strain-energy function. In section 3, it is demonstrated that these assumptions yield an eigenvalue problem for a system differential equation with variable coefficients, which are subsequently simplified by finding a solution with separable variables. Section 4 mainly focuses on the application of *WKB* method to the stability analysis of the eigenvalue problem section 3.Then in section 4 with the aid *WKB* method for system problem the instability this model. The final part of the study discusses the conclusions in conjunction with a discussion of the obtained results and suggestions for future research.

2. Basic equations

Equations describing deformations are widely-known [14]. Consider a circular cylindrical tube of length L in which the radii of inner and outer curved surfaces correspond to A and B, respectively. The undeformed tube is then defined by cylindrical co-ordinates(R, Θ, Z), as follows:

$$A \le R \le B, \qquad 0 \le \Theta \le 2\pi, \qquad 0 \le Z \le L.$$

Now it is assumed that cylindrical co-ordinates (r, θ, z) describe thecylinder in the current configuration where

$$a \le r = r(R) \le b, \qquad \theta = \Theta - \omega t, \qquad z = Z.$$

Here a and b represent the new inner and outer radii, respectively, of the tube, ω denotes the angular velocity of the rotating tube, and t denotes time and this leads to the following expression:

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ISSN: 1791-2377
doi:10.25103/jestr.112.17

$a = \mu_1 A, \qquad b = \mu_2 B,$

where μ_1 and μ_2 denote constants, which with respect to the presentcompression problem, satisfy $0 < \mu_1, \mu_2 < 1$. The material of the tube is assumed incompressible, and this leads to the following expression:

$$r^2 - a^2 = R^2 - A^2$$

Incompressibility implies the absence of a change in volume and thus μ_1 and μ_2 are related as follows:

$${\mu_2}^2 = 1 - (1 - {\mu_1}^2)(A/B)^2$$

Here after, it is assumed that all variables and parameters with length dimensions are scaled by B. Additionally, λ_1, λ_2 and λ_3 denote the principal stretchescorresponding to the (r, θ, z) directions, respectively. This results in the following expression:

$$\lambda_1 = \frac{dr}{dR}, \quad \lambda_2 = \frac{r}{R}, \quad \lambda_3 = 1.$$

With respect to an incompressible isotropic elastic solid, the strain-energy function per unit volume corresponds to $W = W(\lambda_1, \lambda_2, \lambda_3)$and the principal components of the Cauchy stress associated withσ_{ii} such that the following expression holds:

$$\sigma_{ii} = \sigma_i - p, \quad (i = 1,2,3),$$

where

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i}, \quad (i = 1,2,3),$$

and p denotes the arbitrary hydrostatic pressure. The equation ofmotion is reduced as follows:

$$\frac{d\sigma_{11}}{dr} + \frac{1}{r}(\sigma_{11} - \sigma_{22}) = -\rho r \omega^2,$$

Where ρ denotes the uniform density of the material. It is assumed that traction is absent on the lateral surfaces of the cylinder as follows:

$$\sigma_{11}(a) = \sigma_{11}(b) = 0.$$

3. The incremental equations

The position vector $\boldsymbol{x}$ of a material particle in the current configuration is subjected to an increment corresponding to $\dot{\boldsymbol{x}}$. It is assumed that $\dot{\boldsymbol{s}}_{\mathbf{0}}$ is the corresponding increment in the nominal stress in the current configuration$\boldsymbol{s}_{\mathbf{0}}$.
The equation of motion is expressed as follows:

$$div\, \dot{\boldsymbol{s}}_{\mathbf{0}} = \rho \dot{\boldsymbol{x}}_{tt}, \tag{1}$$

Where div denotes the divergence operator relative to $\boldsymbol{x}$, $\dot{\boldsymbol{x}}_{tt}$denotes the incremental acceleration, and the subscript t denotes the materialtime-derivative. The incremental boundary conditions on the surface is as follows:

$$\dot{\boldsymbol{s}}_{\mathbf{0}}{}^T \boldsymbol{n} = \mathbf{0}, \qquad r = a, b,$$

where $\boldsymbol{n}$ denotes the unit outward normal. It is assumed that $\boldsymbol{e_i}, (i = 1,2,3)$ correspond to the unit base vectors for the(r, θ, z) directions, respectively. The incremental displacement vector is as follows:

$$\dot{\boldsymbol{x}} = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 + w\boldsymbol{e}_3,$$

and the incremental acceleration due to the angular speed ω and its increment $\dot{\omega}$ are given as follows:

$$\dot{\boldsymbol{x}}_{tt} = (u_{tt} - 2\omega\, w_t - \omega^2 u)\boldsymbol{e}_1 + (v_{tt} + 2\omega\, u_t - \omega^2 v)\boldsymbol{e}_2 + w_{tt}\boldsymbol{e}_3,$$

where the subscript t denotes differentiation with respect totime. The linearized $\dot{\boldsymbol{s}}_{\mathbf{0}}$ is expressed as follows:

$$\dot{\boldsymbol{s}}_{\mathbf{0}} = \boldsymbol{\mathcal{A}}\, \boldsymbol{\Gamma}^{\mathrm{T}} + p\ \boldsymbol{\Gamma} - \dot{p}\boldsymbol{I},$$

where

$\boldsymbol{\Gamma}, \dot{p}$, and $\boldsymbol{I}$ denote the incrementaldeformation gradient $\boldsymbol{F}$, the increment in p and theidentity tensor, respectively, and $\boldsymbol{\mathcal{A}}$ denotes the fourth-order tensor of fixed-reference elastic moduli are defined [1].
The matrix of components of $\boldsymbol{\Gamma}$ with respect to the abovebasis is as follows:

$$\boldsymbol{\Gamma} = \begin{bmatrix} u_{,r} & u_{,\theta} & u_{,z} \\ v_{,r} & u + v_{,\theta} & v_{,z} \\ w_{,r} & w_{,\theta} & w_{,z} \end{bmatrix}.$$

Additionally, the incremental incompressibility condition is as follows:

$$tr(\boldsymbol{\Gamma}) = 0.$$

With respect to a neo-Hookean material, the strain energy function takes the following form:

$$W = \mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),$$

by simply appending the time-dependent term in equation (1) and the incremental equations for a rotating thick-walled tube were given in an extant study [1]

$$F'''' + \Sigma_3 F''' + \Sigma_2 F'' + \Sigma_1 F' + \Sigma_0 F = 0\,, \tag{2}$$

where $F = F(r)$ corresponds to a prime that denotes differentiation with respect to r and the following expression:

$$\Sigma_3 = \frac{6}{r} + \frac{2\mathcal{A}'_{1111}}{\mathcal{A}_{1111}}, \quad \Sigma_2 = \frac{5\mathcal{A}_{1111} - m^2\mathcal{A}_{1111} - m^2\mathcal{A}_{2222} + 7\, r\mathcal{A}'_{1111}}{r^2\mathcal{A}_{1111}} + \frac{\mathcal{A}''_{1111}}{\mathcal{A}_{1111}},$$

$$\Sigma_1 = -\frac{\mathcal{A}_{1111} + m^2(\mathcal{A}_{1111} + \mathcal{A}_{2222}) - r\mathcal{A}'_{1111} + m^2 r(\mathcal{A}'_{1111} + \mathcal{A}'_{2222})}{r^3\mathcal{A}_{1111}} + \frac{\mathcal{A}''_{1111}}{r\mathcal{A}_{1111}},$$

$$\Sigma_0 = \frac{(-1+m^2)(-\mathcal{A}_{1111}+m^2\mathcal{A}_{2222}+r\mathcal{A}'_{1111})}{r^4\mathcal{A}_{1111}} + \frac{(-1+m^2)\mathcal{A}''_{1111}}{r^2\mathcal{A}_{1111}},$$

where m denotes the longitudinal mode number. Finally, the following expression is subject to the boundary conditions:

$$F''' + \Sigma_{12}F'' + \Sigma_{11}F' + \Sigma_{10}F = 0, \qquad r = a_1, a_2$$

$$F'' + \Sigma_{21}F' + \Sigma_{20}F = 0, \qquad r = a_1, a_2$$

where

$$\Sigma_{10} = \frac{(-1+m^2)\mathcal{A}_{1111} + r(m^2 r\rho\omega^2 + (-1+m^2)\mathcal{A}'_{1111})}{r^3\mathcal{A}_{1111}},$$

$$\Sigma_{11} = \frac{\mathcal{A}_{1111} - m^2(\mathcal{A}_{1111}+\mathcal{A}_{2222}+\sigma_1) + r\mathcal{A}'_{1111}}{r^2\mathcal{A}_{1111}}, \Sigma_{12} = \frac{4}{r} + \frac{\mathcal{A}'_{1111}}{\mathcal{A}_{1111}},$$

$$\Sigma_{20} = \frac{(-1+m^2)\sigma_1}{r^2\mathcal{A}_{1111}}, \qquad \Sigma_{21} = \frac{2\mathcal{A}_{1111} - \sigma_1}{r\mathcal{A}_{1111}}.$$

4. Asymptotic analysis

Following [6], the *WKB* method is applied to derive solutions of the following form:

$$F = \exp\left(\int_{a_1}^{r} \Phi(r)dr\right), \quad \Phi = \Phi_0 + \tfrac{1}{m}\Phi_1 + \tfrac{1}{m^2}\Phi_2 + \cdots, \tag{3}$$

where it is necessary to determine the functions Φ_0, Φ_1,.... For the purpose of this study, it is sufficient to only examine the three leading terms (since $\frac{1}{m^k}$ is very small with respect to $m \gg 1$ and $k \geq 3$). Substituting (3) into the (2) results in the following expression with respect to the leading order in m:
$\mathcal{A}_{1111}r^4{\Phi_0}^4$ - ($\mathcal{A}_{1111}$+$\mathcal{A}_{2222}$) $r^2{\Phi_0}^2$ + $\mathcal{A}_{2222}$ =0.

Generally, this corresponds to quadric equations for Φ_0, and it is expected that four independent solutions are obtained as follows: ${\Phi_0}^{(i)}, i = 1,2,3,4,$

$${\Phi_0}^{(1)} = -{\Phi_0}^{(2)} = \frac{1}{r}, \quad {\Phi_0}^{(3)} = -{\Phi_0}^{(4)} = \frac{1}{r\mathcal{A}_{1111}}.$$

The second order ordinary differential equation for Φ_1 is the same for both repeated roots and produces two independent solutions, and thus, the following expression is derived:

$${\Phi_1}^{(1)} = {\Phi_1}^{(2)} = \frac{\mathcal{A}'_{1111}(1+{\mathcal{A}_{1111}}^2)}{2\mathcal{A}_{1111}(1-{\mathcal{A}_{1111}}^2)},$$

$${\Phi_1}^{(3)} = {\Phi_1}^{(4)} = \frac{\mathcal{A}'_{1111}}{2\mathcal{A}_{1111}(1-{\mathcal{A}_{1111}}^2)}.$$

With respect to the fore-mentioned results, the general solution can be derived as follows:

$$F = \sum_{i=1}^{4} \xi^{(i)} E^{(i)}(r),$$

where

$$E^{(i)}(r) = \exp\left(\int_{a_1}^{r} \Phi^{(i)}(r)dr\right), \quad \Phi^{(i)} = \Phi_0^{(i)} + \frac{1}{m}\Phi_1^{(i)} + \cdots,$$

for some constants $\xi^{(i)}, i = 1,2,3,4.$

The boundary conditions can be expressed as a matrix equation of the Form as follows:

$$\sum_{i=1}^{4} \Omega_{ij}\xi^{(i)}, \quad j = 1,2,3,4,$$

where

$$\Omega_{ij} = \begin{bmatrix} \alpha^{(1)}(a) & \alpha^{(2)}(a) & \alpha^{(3)}(a) & \alpha^{(4)}(a) \\ \beta^{(1)}(a) & \beta^{(2)}(a) & \beta^{(3)}(a) & \beta^{(4)}(a) \\ E^{(1)}\alpha^{(1)}(b) & E^{(2)}\alpha^{(2)}(b) & E^{(3)}\alpha^{(3)}(b) & E^{(4)}\alpha^{(4)}(b) \\ E^{(1)}\beta^{(1)}(b) & E^{(2)}\beta^{(1)}(b) & E^{(3)}\beta^{(1)}(b) & E^{(4)}\beta^{(1)}(b) \end{bmatrix}.$$

The functions $\alpha^{(j)}(r)$ and $\beta^{(j)}(r)$ in the above matrix are defined as follows:

$$\alpha^{(j)}(r) = \left(\Phi^{(i)}(r)\right)^2 + \left(\Phi^{(i)}(r)\right)' + \frac{1}{r}\left(\Phi^{(i)}(r)\right) + \frac{m^2+1}{r^2},$$

$$\beta^{(j)}(r) = \left(\Phi^{(i)}(r)\right)^3 + \left(\Phi^{(i)}(r)\right)'' + \left(3\Phi^{(i)}(r) + \frac{2(k+r^2)}{k\,r(1+r^2)}\right)\left(\Phi^{(i)}(r)\right)'$$

$$+ \frac{2(k+2\,r^2)}{k\,r(1+r^2)}\left(\Phi^{(i)}(r)\right)^2 + \left(\frac{r^2 - 3\,m^2r^2 - 4\,k\,m^2}{(k+r^2)^2} - \frac{k^2(1+2m^2)}{r^2(k+r^2)^2}\right)\left(\Phi^{(i)}(r)\right)$$

$$\left(\frac{k(1-m^2)}{r^3(k+r^2)} + \frac{m^2-1}{r(k+r^2)} + \frac{m^2\,r\,\rho\,\omega^2}{k+r^2}\right).$$

A non-trivial solution for $\xi^{(i)}$ requires $det\,(\Omega_{ij}) = 0$, that corresponds to the required bifurcation condition in the large m limit. However, given that $E^{(i)}(a) = 1$ and $E^{(1)}(b) = E^{(3)}(b)$ are exponentially large whereas $E^{(2)}(b) = E^{(4)}(b)$ are exponentially small, the four equations for $\xi^{(i)}$ can be decoupled into two pairs of equations as indicated in a previous study [10]. Thus, the equations can be reduced as follows:

$$\mu_1 = \gamma_0 + \frac{\gamma_1}{m} + \frac{\gamma_2}{m^2} + \cdots,$$

where

$$\gamma_0 = 0.543689, \quad \gamma_1 = 0.238662(-1.47573 + 0.113207\ \omega_1^2),$$

$$\gamma_2 = 0.238662(15.5584 + 0106799\ \omega_1^2 - 0.0206285\ \omega_1^4).$$

where $\omega_1 = A\omega\sqrt{\rho}$.

With respect to $\omega_1 = 0$, this fact is confirmed by results obtained in an extant study [1]. Table 1 illustrates μ_1 for a given ω_1 and mode number m. It should be noted that with respect to all the values denoted by ω_1, present compression μ_1 is obtained for different mode numbers.

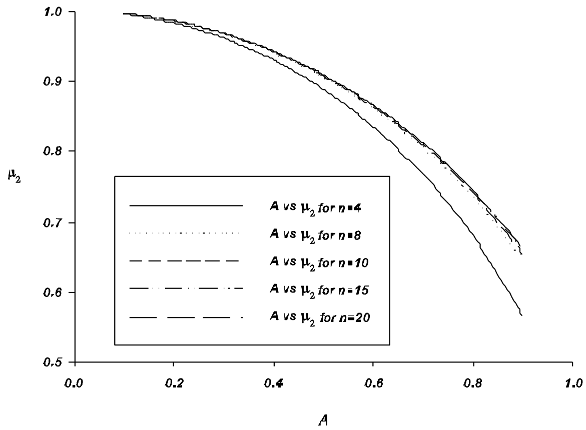
Table 1. The present compression μ_1 on the boundary conditions with respect to ω_1 for $m = 4,8,10,15,20$.

ω_1	$\mu_1(m = 4)$	$\mu_1(m = 8)$	$\mu_1(m = 10)$	$\mu_1(m = 15)$	$\mu_1(m = 20)$
0.2	0.3980	0.5289	0.5412	0.5503	0.5517
0.4	0.3964	0.5282	0.5406	0.5499	0.5515
0.6	0.3948	0.5275	0.5400	0.5495	0.5512
1	0.3919	0.5260	0.5389	0.5488	0.5506
1.5	0.3878	0.5242	0.5375	0.5478	0.5499
2	0.3842	0.5225	0.5361	0.5469	0.5492

Table 2 lists μ_2 for various A mode numbers m and $\omega_1 = 1$. In addition to the observations as shown in the table an upper bound μ_2 is likely to exist and corresponds to $0.6553.\ m \to \infty, A \to 1$).

Table 2. The present compression μ_2 on the boundary conditions with respect to A for $m = 4,8,10,15,20$.

A	$\mu_2(m = 4)$	$\mu_2(m = 8)$	$\mu_2(m = 10)$	$\mu_2(m = 15)$	$\mu_2(m = 20)$
0.2	0.9829	0.9854	0.9857	0.9859	0.9860
0.3	0.9611	0.9669	0.9675	0.9680	0.9681
0.5	0.8879	0.9051	0.9070	0.9085	0.9087
0.7	0.7650	0.8035	0.8076	0.8109	0.8115
0.9	0.5606	0.6435	0.6521	0.6587	0.6600



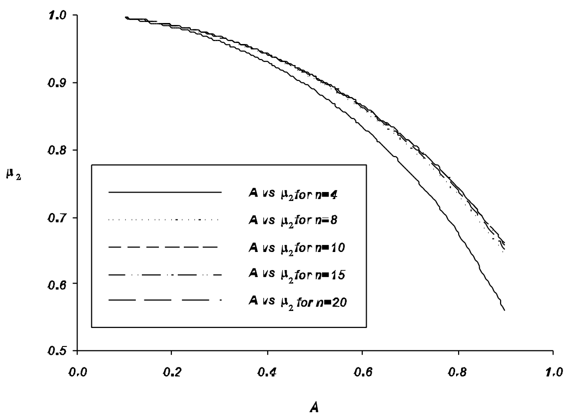


Fig. 1. Plot of the values of μ_2 with respect to A for $\omega_1 = 0$, (left), $\omega_1 = 1$, (right) and $m = 4,8,10,15,20$.

5. Concluding remarks

This study examined the deformation in rotating thick-walled circular cylinders of incompressible isotropic neo-Hookean material. First, a system of differential equations with related boundary conditions was derived. Second, it was demonstrated that the WKB method could be typically applied to obtain a first-order approximation of the bifurcation criterion. The results indicated that the dependence of μ_1 involves a boundary layer structure. Additionally, simple asymptotic expressions for the bifurcation condition were obtained.

Furthermore, a higher-order asymptotic expansion was obtained, and the deduced expansion provided guidance in deriving the correct approximation for μ_2. Mode numbers tend to infinity, and thus, in the special case corresponding to ω_1, the obtained outer layer approximately corresponded to $0.5651(m \to \infty, A \to 1)$.

Acknowledgment:

The author thanks Professor Erasmo Carrera in Politecnico di Torino.

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